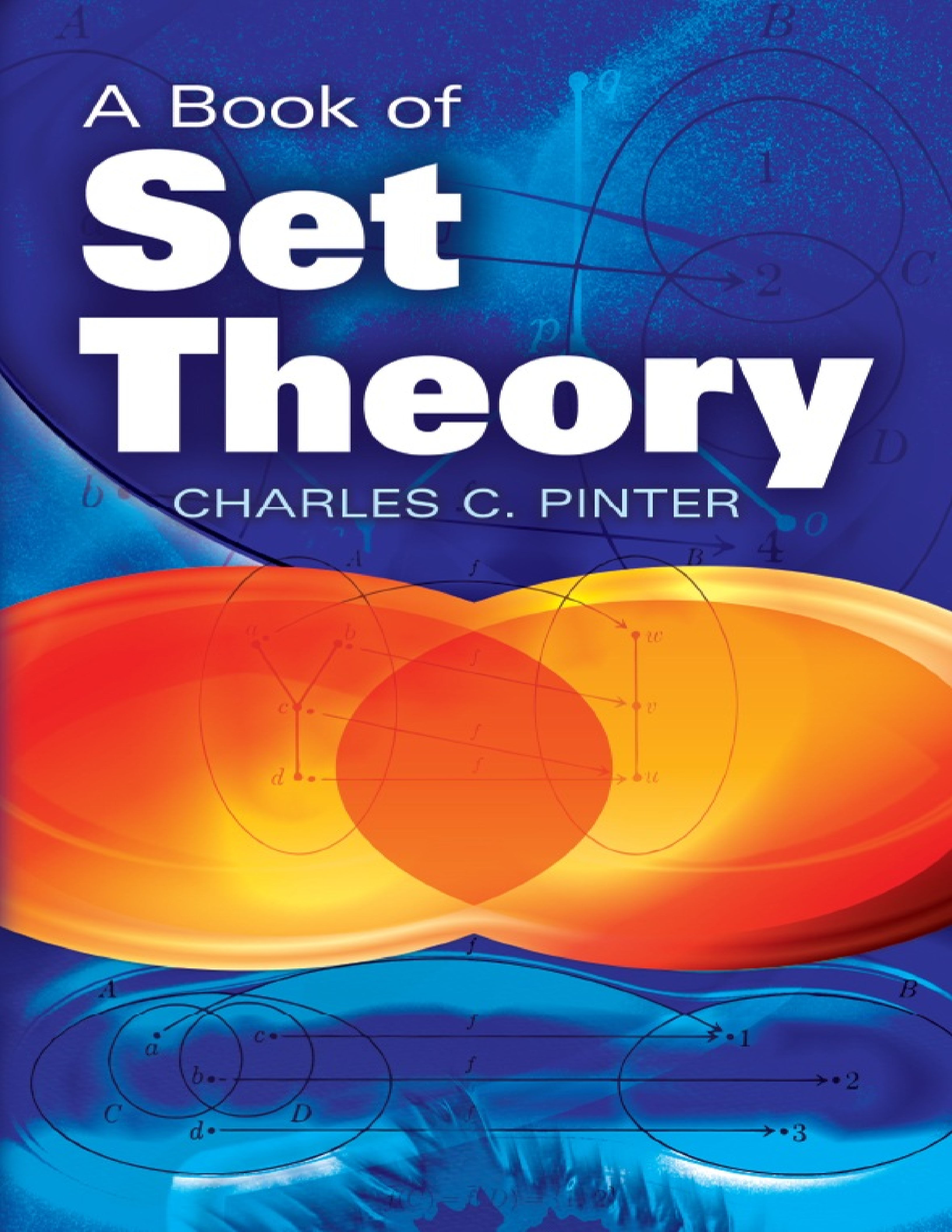
A Book of Set Theory

CHARLES C. PINTER



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SET THEORY

CHARLES C. PINTER

Emeritus Professor

Bucknell University

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Summary: "This accessible approach to set theory for upper-level undergraduates poses rigorous but simple arguments. Each definition is accompanied by commentary that motivates and explains new concepts. A historical introduction is followed by discussions of classes and sets, functions, natural and cardinal numbers, the arithmetic of ordinal numbers, and related topics. 1971 edition with new material by the author"—Provided by publisher.

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To my students,
from whom I have learned how to explain and how to teach

Contents

Preface

Over many years of selecting instructional materials for my courses, I came to understand that textbooks that are congenial to students obey a law of reciprocity that I wish to propose as an axiom:

Axiom The more effort an author puts into writing a text, the less effort is required of the reader to understand it.

I adopted this guiding principle as a categorical imperative during the writing of this book. Even after all the mathematics was in place, I reread and rewrote many times and tested explanatory strategies with my students. I have dedicated this book to them because they have been my most patient and honest critics. The book owes its present form largely to them and to their (sometimes naïve, often brilliant) suggestions.

Mathematics has a superbly efficient language by means of which vast amounts of information can be elegantly expressed in a few formal definitions and theorems. It is remarkable that the life work of consecutive generations of great thinkers can often be summed up in a set of equations. The economy of the language masks the richness and complexity of the thoughts that lie behind the symbols. Every mathematics student has to master the conventions for using its language effectively. However, what is far more important is that the student be initiated into the inner life of mathematics—the images, the intuitions, the metaphors that, once grasped, make us say, "Aha! Now I understand! Now I *see* it!" This inner seeing is what makes mathematics vital and exciting.

What is most unique about set theory is that it is the perfect amalgam of the visual and the abstract. The notions of set theory, and the ideas behind many of the proofs, present themselves to the inner eye in vivid detail. These pictures are not as overtly visual as those of geometry or calculus. You don't see them in the same way that you see a circle or a tangent line. But these images are the way into abstraction. For the maturing student, the journey deeper into abstraction is a rite of passage into the heart of mathematics.

Set theory is also the most "philosophical" of all disciplines in mathematics. Questions are bound to come up in any set theory course that cannot be answered "mathematically", for example with a formal proof. The big questions cannot be dodged, and students will not brook a flippant or easy answer. Is the continuum hypothesis a fact of the world? Is the axiom of choice a truth? If we cannot answer with a definite *yes* or *no*, in what manner are they justified? (That one requires a long answer). What is the meaning of the Jacob's ladder of successive infinities so high that the very thought of it leads to a kind of intellectual vertigo? To what extent does mathematics dwell in a Platonic realm—and if it does, then in the words of Eugene Wigner, "how do you explain the unreasonable effectiveness of mathematics in the natural world?" Or as Descartes wondered, where do you find the nexus between the material world and the products of thought?

In this book I have tried, insofar as possible, not to evade these questions nor to dwell on them excessively. Students should perceive that mathematics opens doors to far-reaching and fascinating questions, but on the other hand they must remain anchored in mathematics and not get lost in the narcotic haze of speculative thought. I have tried to provide the necessary background for understanding axiomatics and the purpose and meaning of each of the axioms needed to found set theory. I have tried to give a fair account of the philosophical problems that lie at the center of the formal treatment of infinities and other abstractions. Above all, of course, I have endeavored to present the standard topics of set theory with uncompromising rigor and precision, and made it clear that the formalism on the one hand, and the intuitive explanations on the other hand, belong in two separate domains, one useful for understanding, the other essential for *doing* mathematics.

This book is a revised and re-written version of an earlier edition, published in 1972 by Addison-Wesley. I have retained most of the formal definitions, theorems and proofs, with nothing more than a few corrections where needed. I have also retained the initial chapter that narrates the origins and early history of set theory, because history (in general) does not change. I have added commentary, introduced some new discussions, and reorganized a few proofs in order to make them cleaner and clearer. Finally, I have added a new chapter on models of set theory and the independence results of Gödel and Cohen. I have set the discussion of these topics at a level that is accessible to undergraduates while not concealing the difficulties of the subject.

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Historical Introduction

1 THE BACKGROUND OF SET THEORY

Although set theory is recognized to be the cornerstone of the “new” mathematics, there is nothing essentially new in the intuitive idea of a set. From the earliest times, mathematicians have been led to consider sets of objects of one kind or another, and the elementary notions of modern set theory are implicit in a great many classical arguments. However, it was not until the latter part of the nineteenth century, in the work of Georg Cantor (1845–1918), that sets came into their own as the principal object of a mathematical theory.

Strangely, it was his work in the highly technical field of trigonometric series which first led Cantor to study the properties of sets. At first, he confined himself to certain particular sets of real numbers which occurred in connection with the convergence of series. But Cantor was quick to understand that his discoveries applied to sets quite generally; in a series of remarkable papers, published between 1873 and 1897, he moved progressively further from the concrete problems which had initiated his thinking on sets, and toward the powerful general concepts which underlie set theory today.

The boldest step which Cantor had taken—in the eyes of his contemporaries–was his use of infinite sets, which he considered as no less natural than using finite sets. The question of “infinity” had long been one of the most sensitive problems of mathematics. The reader is undoubtedly acquainted with Zeno’s famous “paradox”, in which a unit line segment is divided into subintervals by the points 1/2, 1/4, 1/8, 1/16, etc. Each subinterval—no matter how small—has a definite, nonzero length, and there are infinitely many subintervals; hence, the seemingly paradoxical conclusion that infinitely many nonzero lengths can be added together to produce a finite length. In order to avoid such traps, classical mathematicians made a distinction between the “actual” infinite—in which infinitely many objects are conceived of as existing simultaneously—and the “virtual” infinite, which is simply the potential to exceed any given finite quantity. The “virtual” infinite was regarded as safe, hence admissible, whereas the “actual” infinite was taboo.

It is not surprising then, that Cantor’s theory—with its uninhibited use of infinite sets (the notion of infinite was obviously understood here in the “actual” sense)—was not immediately accepted by his contemporaries. It was received at first with skepticism, sometimes even with open hostility. However, by the 1890’s the more “palatable” parts of Cantor’s theory were widely used, for they provided an elegant framework for a wide variety of mathematical theories. And before the turn of the century, even the most revolutionary aspects of set theory had been accepted by a great many mathematicians—chiefly because they turned out to be invaluable tools, particularly in analysis.

Meanwhile, the work of several outstanding mathematicians, in particular Dedekind, was taking a turn which would cast set theory in its most promising role—as the fundamental, “unifying” branch of mathematics. From the earliest times, mathematicians have given thought to the possibility of unifying the entire discipline under a small number of basic principles. Many of the ancient schools, from Euclid through the Middle Ages, contended that the various branches of mathematics could be subsumed under geometry (numbers might be conceived as geometric proportions); a far more successful attempt at unification came in the nineteenth century, when the work of Weierstrass, Dedekind, and others suggested that all of classical mathematics could be derived from the arithmetic of the natural numbers (positive integers). It was shown that every real number can be regarded as a sequence (called a "Cauchy sequence") of rational numbers; hence the study of the real numbers is reduced to that of the rational numbers. But the rational numbers can easily be regarded as pairs of integers, so finally the mathematics of the real numbers—which includes the calculus and (via analytic geometry) all of geometry—can be based on the natural numbers.

It was at this crucial point in the evolution of ideas on the foundations of mathematics that Dedekind, in his little book *Was sind und was sollen die Zahlen* (1888), revealed that the concept of natural numbers can be derived from the basic principles of set theory. A modern way to show this is the following: we let "0" be the empty set (that is, the set with no elements, denoted by the symbol Ø); "1" is defined to be the set {Ø}, that is, the set (of sets) containing the one element Ø. Then, "2" is defined to be the set {0, 1}, "3" is defined to be {0, 1, 2}, and so on. All the properties of the natural numbers can be proven using these definitions and elementary set theory.

By the turn of the century, then, set theory had not only been accepted as an indispensable tool by a large segment of the mathematical community, but, moreover, it was a serious contender for the position of primacy among the mathematical sciences.

Ironically, at the very time when Cantor's ideas seemed finally to have gained acceptance, the first of certain "paradoxes" were announced, which eventually cast serious doubts as to the basic soundness of set theory in its "Cantorian" form. These paradoxes had such wide repercussions that it is worth looking at them in some detail.

2 THE PARADOXES

Between 1895 and 1910 a number of contradictions were discovered in various parts of set theory. At first, mathematicians paid little attention to them; they were termed "paradoxes" and regarded as little more than mathematical curios. The earliest of the paradoxes was published in 1897 by Burali-Forti, but it had already been discovered, two years earlier, by Cantor himself. Since the Burali-Forti paradox appeared in a rather technical region of set theory, it was hoped, at first, that a slight alteration of the basic definitions would be sufficient to correct it. However, in 1902 Bertrand Russell gave a version of the paradox which involved the most elementary aspects of set theory, and therefore could not be ignored. In the ensuing years other contradictions were discovered, which seemed to challenge many of the "safest" notions of mathematics.

The "paradoxes" of set theory are of two different kinds, the one called *logical* paradoxes, the other called *semantic* paradoxes. The reason for the names "logical" and 'semantic" will become clear to us when we have seen a few examples of these paradoxes; essentially, the "logical" paradoxes arise from faulty logic whereas the "semantic" paradoxes arise from the faulty use of language.

We will devote the remainder of this section to the presentation of two of the most celebrated paradoxes, which involve only elementary concepts of set theory. The first is a "logical" paradox, the second is a "semantic" paradox; both may be considered as typical of their kind.

The simplest of the logical paradoxes is *Russell's paradox*, which can be described as follows:

> If *A* is a set, its elements may themselves be sets; this situation occurs frequently in mathematics—for example, *A* may be a set of lines, where each line is regarded as a set of points. Now the possibility arises that *A* may be an element of itself; for example, the set of all sets has this property.
>
> Let *S* denote the *set of all sets that are not elements of themselves*. Is *S* an element of itself? Well, if *S is* an element of *S*, then—by the very definition of *S*—*S* is *not* an element of *S*. If *S* is

> *not* an element of *S*, then (again, because of the way *S* is defined) *S is* an element of *S*. Thus, we have proven that *S* is an element of *S* if and only if *S* is not an element *S*—a contradiction of the most fundamental sort.

Usually, in mathematics, when we reach a contradiction of this kind, we are forced to admit that one of our assumptions was in error. In this case, we are led to conclude either that it is meaningless to speak of a set as being an element of itself, or that there is no such thing as a "set of all sets which are not elements of themselves." We will return to this question presently; meanwhile, let us say a few words about the semantic paradoxes.

> Typical of the semantic paradoxes is *Berry's paradox*:

> For the sake of argument, let us admit that all the words of the English language are listed in some standard dictionary. Let *T* be *the set of all the natural numbers that can be described in fewer than twenty words of the English language*. Since there are only a finite number of English words, there are only finitely many combinations of fewer than twenty such words—that is, *T* is a finite set. Quite obviously, then, there are natural numbers which are greater than all the elements of *T*; hence there is a *least natural number which cannot be described in fewer than twenty words of the English language*. By definition, this number is not in *T*; yet we have described it in sixteen words, hence it is in *T*.

Once again, we are faced with a glaring contradiction; since the above argument would be unimpeachable if we admitted the existence of the set *T*, we are irrevocably led to the conclusion that a set such as *T* simply cannot exist.

Before the paradoxes, the question of the *existence* of sets had never been posed. Cantor "defined" a set to be "a collection of definite distinguishable objects of our perception which can be conceived as a whole." More specifically, Cantor and his early followers accepted the "common-sense" notion that if we can describe a property of objects, we can also speak of the set of all objects possessing that property. The paradoxes had the singular merit of proving this native conception of sets to be unacceptable—if only because certain "properties" lead to paradoxical sets.

In the various movements which sprang up, during the early 1900's, with the aim of revising the foundations of set theory, the topic of central concern was the *existence* of sets. What properties legitimately defined sets? Under what conditions do properties define sets at all? How can new sets be formed from existing ones?

3 THE AXIOMATIC METHOD

The appearance of the paradoxes marked the beginning of a crisis in the foundations of mathematics which has not been completely resolved to our day. It became abundantly clear that the intuitive conception of a set, as embodied in Cantor's "definition," does not provide a satisfactory basis for set theory—much less for mathematics as a whole. Minor attempts to eliminate the paradoxes by excluding specific types of concepts and definitions were doomed to failure; nothing less than an entirely new approach was needed. Starting about 1905, several ways of dealing with the problem were proposed and developed by their adherents; most of them can be classified into three major groups, called the "axiomatic," the "logistic," and the "intuitionist" schools. The remainder of this chapter is devoted to presenting these three ways of thought. First, however, we shall briefly review the development of the axiomatic method.

The axiomatic method in mathematics emerged in a highly developed form, about 300 B.C., with the appearance of Euclid's *Elements*. Although the method popularized by Euclid has become a characteristic feature of every branch of mathematics today, only in comparatively recent years has it been applied outside of geometry. For this reason, our modern understanding of axiom systems, and of deductive reasoning generally, has to a great extent come out of studies in the field of geometry. It is worth examining a few of the major developments in geometry which influenced the growth of the axiomatic method.

To Euclid and his times, the axioms and postulates represent "truths" whose validity is beyond question. For example, it was this belief in the absolute truth of geometric propositions which led to the millenia-long controversy over Euclid's "parallel postulate." This postulate asserts that if two lines, *A* and *B*, intersect a third line *C*, and if the interior angles which *A* and *B* make with *C* (on a given side of *C*) add up to less than two right angles, then *A* and *B* necessarily intersect. Because this statement appeared to be "obviously true"—yet it lacked the terse simplicity of the other axioms and postulates—geometers from Euclid to the 1700's succeeded one another in vain attempts to prove it from the remaining assumptions. Only in the midnineteenth century was the question resolved when Bolyai and Lobachevski, each replacing the parallel postulate by an assumption which *contradicted* it, developed "non-Euclidean" geometries. The non-Euclidean geometries were shown to be no less consistent than Euclidean geometry, since they could be given Euclidean interpretations (that is, by suitably reinterpreting "point," "line," "angle," and so forth, the postulates of either Bolyai or Lobachevski can be made to hold in Euclidean geometry). Thus, not only is the parallel postulate independent of the other axioms and postulates of Euclid's system, but alternative, equally consistent geometries can be founded which do not describe the space of our everyday experience. With this came the recognition that axioms are not "universal truths," but are whatever statements we wish to use as premises in an argument.

Perhaps the greatest defect in the *Elements* is the number of tacit assumptions made by Euclid—assumptions not granted by the postulates. For example, in a certain proof it is assumed that two circles, each passing through the center of the other, have a pair of points in common—yet the postulates do not provide for the existence of these points. Elsewhere, Euclid speaks of a point as being *between* two others, yet he does not define "betweenness' or postulate any of its properties. Other arguments in the *Elements* involve the concept of rigid motion—a concept which is not defined or mentioned in the postulates. Thus, throughout Euclid, the orderly chain of logical inferences is frequently broken by tacit appeals to visual evidence. With the discovery of these gaps, mainly in the nineteenth century, grew the understanding that a mathematical argument must be able to proceed without the mediation of spatial or other intuition; that certain objects and relations (such as "point," "line," "betweenness") must be regarded as *undefined notions* and their properties fully specified; that deduction is, in a very essential manner, independent of the *meaning* of concepts. In 1882, M. Pasch published the first formulation of geometry in which the exclusion of any appeal to intuition is clearly stated as a goal and systematically carried out.

By the end of the nineteenth century, then, a modern conception of the axiomatic method began to emerge. In its broad outlines, it did not differ from the ideas held by Euclid: a mathematical theory is "axiomatic" if certain statements are selected to be "axioms," and all the remaining propositions of the theory are derived from the axioms by logical inference. However, there was a new understanding of the *formal* nature of mathematical proof. Inasmuch as possible, the axioms should be sufficiently detailed, and the rules of logical deduction sufficiently explicit, that neither intuition nor intelligence is needed to go through the steps of a proof. Ideally, it should be possible for a computer to verify whether or not a proof is correct.

As long as mathematics is formulated in ordinary languages, such as English, human understanding is indispensable for interpreting statements and finding the structure of complex sentences. Thus, if intuition is to be completely removed from mathematical proof, an essential prerequisite is the development of a *formal* mathematical language: the “rules” of this language must be strictly codified, so that every statement is unambiguous and its structure clear. The creation of formal, symbolic languages was one of the most important developments of modern mathematics; here is what such a language looks like.

The most basic mathematical statements look like this:

“X is parallel to Y,”
“y lies between x and z,”
“X is an open set,” etc.

They are statements about an object, a pair of objects, or more generally, about an ordered n-tuple of objects. These statements are called *elementary predicates*, and the letters X, Y, x, y, z are called their *variables*.

It is convenient to denote a predicate by a single letter followed by the list of its variables. Thus, “X is parallel to Y ” may be written $A(X, Y)$, “y lies between x and z” may be written $B(x, y, z)$, and so on. Now the mature student is aware of the fact that the “meaning” of a predicate is immaterial in the process of mathematical reasoning. For example, the “meaning” of the word *parallel* has no bearing on the course of a geometrical argument; all that matters is the relationship between the statement “X is parallel to Y ” and other statements such as “X intersects V ” and “Y is perpendicular to Z.” For this reason, elementary predicates are also called *atomic formulas*; they are integral, “indivisible,” not to be analyzed further; they are only to be distinguished from each other.

It is a remarkable fact that every known branch of mathematics requires only a finite number (usually a very small number) of distinct elementary predicates. For example, every statement of plane Euclidean geometry can be expressed in terms of the following basic predicates:

	$P(x)$:	x is a point.
	$L(x)$:	x is a line.
	$B(x, y, z)$:	y is between x and z.
1)	$E(x, y)$:	x equals y.
	$I(x, y)$:	x belongs to y.
	$C(u, v, x, y)$:	the segment uv is congruent to the segment xy.
	$D(u, v, w, x, y, z)$:	the angle uvw is congruent to the angle xyz.

Set theory, as we shall see, may be formulated entirely in terms of the one predicate $x \in A$ (x is an element of A).

Predicates alone are not sufficient to express all the statements of mathematics, just as nouns alone would be inadequate to write English sentences. For example, we may wish to say that *if* “x is parallel to y” *and* “y is perpendicular to z,” *then* “x is perpendicular to z.” Such statements consist of predicates joined together by means of *logical connectives*. Thus, if P and Q are statements in our language, then so are the following:

$$\begin{array}{rl} \neg P: & \text{not } P. \\ P \wedge Q: & P \text{ and } Q. \\ P \vee Q: & P \text{ or } Q. \\ P \Rightarrow Q: & P \text{ implies } Q. \\ P \Leftrightarrow Q: & P \text{ if and only if } Q. \end{array}$$

Finally, we may wish to say, for example, that if “*x* is a point” and “*y* is a point,” then *there exists* a point *z* such that “*z* is between *x* and *y*.” This requires the use of quantifiers. Thus, if *P*(*x*) is a statement with a variable *x*, then the following are also statements:

$$\begin{array}{rl} \forall x,\, P(x): & \text{for every } x,\ P(x). \\ \exists x \ni P(x): & \text{there exists an } x \text{ such that } P(x). \end{array}$$

This completes our formal mathematical language. All of known mathematics can be expressed in terms of elementary predicates, logical connectives, and quantifiers. To illustrate how this language is used, let us take a simple example. The sentence

“If *x* and *y* are distinct points, then there is a point *z* between *x* and *y*”

can be symbolized as

2) $$[P(x) \wedge P(y) \wedge \neg E(x, y)] \Rightarrow [\exists z \ni (P(z) \wedge B(x, z, y))],$$

where the meaning of the predicates is given in (1) above.

One of the many benefits to be derived from the use of a formal language is that it is possible to described precisely and explicitly the process of deduction in this language. A few clear, unambiguous rules decide when a statement *T* may be inferred from a statement *S*. A few such rules are the following:

3) $$\begin{array}{ll} \text{Rule A:} & \text{from } P \text{ and } P \rightarrow Q \text{ we may infer } Q. \\ \text{Rule B:} & \text{from } P \text{ and } Q \text{ we may infer } P \wedge Q. \\ \text{Rule C:} & \text{from } \neg(\neg P) \text{ we may infer } P. \\ \text{Rule D:} & \text{from } P(c) \text{ we may infer } \exists x \ni P(x). \end{array}$$

These, and a few other laws, * are called “rules of inference” in our language. A *formal* argument, from given premises, is a sequence of expressions of the formal language, where each expression is either a premise, or is derived from a preceding expression (or expressions) by applying one of the rules of inference.

Example Consider a formal language with a predicate *L*(*x*, *y*); let us agree to write $x < y$ for *L*(*x*, *y*). The following is a very simple formal argument in this language.

Premises

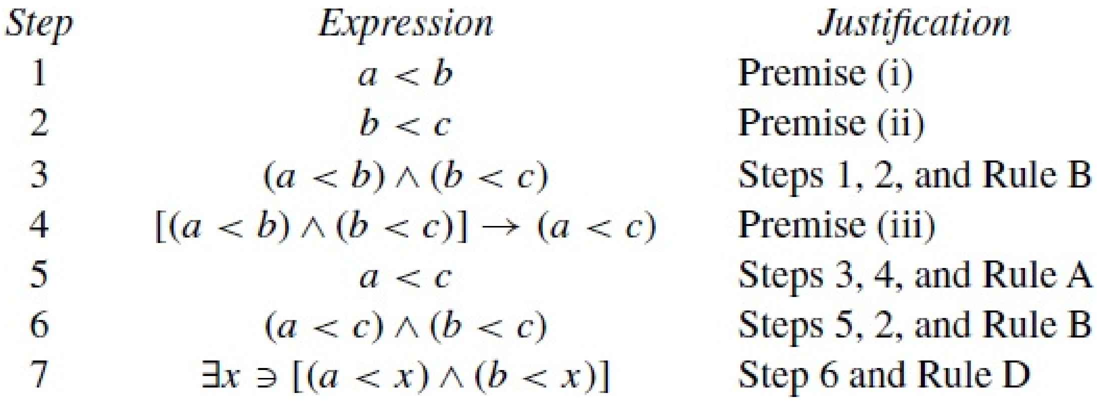
i) $a < b$.

ii) $b < c$.

iii) $[(a < b) \wedge (b < c)] \rightarrow (a < c)$.

Theorem $\exists x \ni [(a < x) \wedge (b < x)]$.

Proof.



Step	*Expression*	*Justification*
1	$a < b$	Premise (i)
2	$b < c$	Premise (ii)
3	$(a < b) \wedge (b < c)$	Steps 1, 2, and Rule B
4	$[(a < b) \wedge (b < c)] \rightarrow (a < c)$	Premise (iii)
5	$a < c$	Steps 3, 4, and Rule A
6	$(a < c) \wedge (b < c)$	Steps 5, 2, and Rule B
7	$\exists x \ni [(a < x) \wedge (b < x)]$	Step 6 and Rule D

The reader should note that the rules of inference are applied to expressions in a perfectly mechanical way. To all intents and purposes, the expressions can be regarded as meaningless arrays of symbols; the fact that they have a meaning *to us* in irrelevant to the task of carrying out the proof. Thus intuition is totally absent from a formal mathematical proof.

An axiomatic theory is said to be *formalized* if its axioms are transcribed in the formal language (for example, formula (2) on p. 8 is an axiom of Hilbert's plane geometry), and all of its proofs are formal proofs. While it is commonly accepted as the ideal, today, that every axiomatic theory be developed formally, it would be far too tedious, in practice, to do so. Symbolic statements are difficult to decipher, and formal proofs tend to be exceedingly long. Thus *mathematicians are usually content to satisfy themselves that an axiomatic theory can be formalized, and then proceed to develop it in an informal manner*. This will be our procedure in this book.

4 AXIOMATIC SET THEORY

To a great many mathematicians in the early 1900's, the answer to the problem posed by the paradoxes was to provide set theory with an axiomatic basis. The term "set" and the relation "is an element of" would be the undefined notions of such a theory, just as "point" and "line" are undefined notions in geometry; their "meaning" would be irrelevant, and their properties would be given formally by the axioms. In particular, *the axioms would be chosen in such a manner that all the useful results of Cantor's theory could be proven, whereas the paradoxes could not.*

The first axiomatization of set theory was given in 1908 by Zermelo. Zermelo's system, with certain modifications due to Skolem and Fraenkel, is widely used up to the present day. Zermelo wrote his work before the time when formal methods became widely understood and accepted; thus his set theory is not written in a formal language, but is closer in style to the older axiomatic treatments of geometry.

In Zermelo's system, there is one primitive relation, denoted by the symbol $\in$; the expression $x \in Y$ is to be read "x is an element of Y". The variables x, y, z, X, Y, etc., which we place to the right or to the left of the symbol $\in$, stand for objects which we agree to call "sets."

The reader may feel there ought to be *two* kinds of objects, namely sets and elements. Actually, this distinction is unnecessary: For, on the one hand, the relationship between element and set is a relative one rather than an absolute one (in fact, the element-set relationship is precisely the relation $\in$). On the other hand, almost every set in mathematics is a set *of sets*. For example, in plane analytic geometry, a line is a set of points; a point is a pair of real numbers (its coordinates); a real number is regarded as a sequence (that is, a set) of rational numbers; etc. Thus a useful simplification which is made in axiomatic set theory is to regard the elements of every set to be sets themselves; in other words, every set is considered to be a set *of sets*. This simplification has no harmful effects, and has the merit of reducing the number of primitive notions and axioms of set theory.

This suggests a comment on notation. Although it is customary to use small and capital letters as in $x \in Y$, it is in no way necessary. In fact, we will sometimes write things like $x \in Y$ and $X \in Y$. All the variables in these expressions denote sets.

Almost every set that arises in our thinking is a set consisting of all the objects of a specified kind—that is, consisting of all the objects which satisfy a given condition. This is the most natural way in which sets occur: we are able to describe a condition on x—let us symbolize this condition by $S(x)$—and we are led to speak of the set of all objects x which satisfy $S(x)$.

Examples

The set of all objects x which satisfy the condition "x is an irrational number and $0 \leqslant x \leqslant 1$" (loosely speaking, the set of all irrational numbers between 0 and 1).

The set of all objects x which can be described by the sentence "x is a man" (loosely speaking, the set of all men).

Since this is the most natural way that sets arise, it is clearly desirable to have a principle in set theory which makes it possible—given any condition $S(x)$—to form the set of all objects x which satisfy $S(x)$. However, as we noted in Section 2, if such a principle is adopted *without any restrictions*, we are led to the paradoxes (for example, we can form the set of all sets which are not elements of themselves). Thus, we must devise such restrictions on this principle as will eliminate the paradoxes. Zermelo conceived the following restriction: Let $S(x)$ be a condition on x; we *cannot* form the set of all x which satisfy $S(x)$; but, if A is a given set, we can form the *set of all x in A which satisfy* $S(x)$. Thus, roughly speaking, a property of objects cannot be used to form a " new" set, but only to " select," from a set A whose existence has already been secured, all the elements which satisfy the given property.

Zermelo introduced this principle as an axiom in his system. Because its role is to *select* elements in sets, he called it the *axiom of selection* and stated it as follows:

> Let A be a set, and let $S(x)$ be a statement about x which is meaningful for every object x in A. There exists a set which consists of exactly those elements in x and A which satisfy $S(x)$.

The set whose existence is give by the axiom of selection is customarily denoted by

$$\{x \in A : S(x)\}$$

[to be read: "the set of all x in A such that $S(x)$"]. Thus the reader should note that Zermelo's system does not allow us to form $\{x : S(x)\}$, [the "set of all x which satisfy $S(x)$"]; but, for any set A, we can form $\{x \in A : S(x)\}$.

How does the axiom of selection avoid the paradoxes? First, let us see what happens to Russell's paradox: the crucial set in Russell's argument is the "set of all sets which are not elements of themselves," which can be symbolized as $\{x : x \notin x\}$. As we have noted, this set cannot be formed in Zermelo's system; the best we can do is to produce $\{x \in A : x \notin x\}$, where A is any set which can be shown to exist. If we substitute $\{x \in A : x \notin x\}$ for $\{x : x \notin x\}$ in Russell's argument, the outcome changes completely. Indeed, let us go through the steps of the argument, with S denoting $\{x \in A : x \notin x\}$:

> $S \in S$ is impossible, for $S \in S$ implies $S \notin S$, a contradiction! Thus $S \notin S$. It follows that $S \notin A$, for if S were in A, then (because $S \notin S$) we would have $S \in S$, which would be a contradiction.

Hence Russell's argument merely proves that if A is any set, then the set $\{x \in A : x \notin x\}$ cannot be an element of A.

The other logical paradoxes disappear in similar fashion. The crucial sets in all of the logical paradoxes have a common trait: they are overly comprehensive—that is, they are "too large," they include too much. In Russell's paradox it is the "set of all sets which are not elements of themselves"; in Cantor's paradox (which is closely related to that of Russell) it is the "set of all sets." The axiom of selection cannot contribute to the formation of these "excessively large" sets, since it can only be used to form *subsets of existing sets*.

The problem of avoiding the semantic paradoxes is a more difficult one. The crucial sets in paradoxes such as Berry's are not "too large." The trouble seems, rather, to be inherent in the condition $S(x)$ which determines the set; even the restriction imposed by the axiom of selection is not an effective barrier. Thus, if $S(x)$ designates the sentence "x can be described in fewer than twenty words of the English language," then the offending set in Berry's paradox is $\{x \in N : S(x)\}$, where N denotes the set of the natural numbers. This set *can* be formed in Zermelo's system, if we admit $S(x)$ as an acceptable condition on x.

Thus, to prevent the semantic paradoxes, we must place restrictions on the type of "conditions" $S(x)$ which are admissible for determining sets. Zermelo attempted to do this by stipulating, in the axiom of selection, that $\{x \in A : S(x)\}$ can be formed only if $S(x)$ *is meaningful for every element x in A*. However, in so doing, he only raised new questions: How are we to understand "meaningful"? How do we determine whether $S(x)$ is meaningful?

We are forced, at last, to face a question which the alert reader may already have asked himself: What do we mean by a "condition" $S(x)$, by a "statement about an object x"? We cannot be content to regard the concept of a "statement about x" as intuitively known, since our purpose now is to axiomatize set theory, that is, to free it of all dependence on intuition. Zermelo failed to give a satisfactory answer to this question, because he did not frame his system in a formal language. However, in 1922, Skolem and Fraenkel, both working on formal axiomatizations of set theory, saw the natural way out of the dilemma: a "statement about x" is simply a statement *in the formal language* with one "free" variable x. (We say that x is *free* in $S(x)$ if x is not governed by a quantifier $\exists x$ or $\forall x$; thus, in $\exists y \ni (x < y)$, x is free whereas y is not).

In Zermelo's system there is only one elementary predicate, denoted by the symbol $\in$. Thus a statement in the formal language is an expression which can be written using only predicates $x \in Y$, $u \in V$, etc., logical connectives, and quantifiers.

If we restrict the "statements" $S(x)$ which can be used in the axiom of selection to those which are expressible in the formal language, we immediately eliminate all the semantic paradoxes. For example, there is no way of writing the sentence "x can be described in fewer than twenty words of the English language" in terms of the formal language. This solution—this way of avoiding the semantic paradoxes—is acceptable from the mathematical point of view, though it is hardly an ideal solution philosophically. Mathematically, we can still form all the sets essential for mathematics; from a broader point of view, though, we cannot form anything like the "set of all men," the "set of all Latin verbs," etc. No better solution has been devised to this day.

We have seen how Zermelo's system, with modifications due to Skolem and Fraenkel, manages to avoid the paradoxes. The remaining axioms of Zermelo's system are similar to those which will be developed in the following chapters. Essentially, they provide for the existence of the set of all the natural numbers (from which we can construct the other number systems of mathematics), and guarantee the existence of unions, intersections, and products of sets. Before going on, we will briefly review another way of axiomatizing set theory, which is of increasing interest in our day; the essential ideas are due to von Neumann.

Von Neumann noted that two facts combine to produce the logical paradoxes: in the first place, as we have seen, the crucial sets (for example, the set S in Russell's paradox) are "too large;" in the second place, these "large" sets *are allowed to be elements of sets* (for example, it is admitted that Russell's set S may be an element of itself). Of these two facts, Zermelo used the first; he avoided the paradoxes by making it impossible to form the 'large" sets. Von Neumann proposed to use the second of these facts: he would permit the excessively large sets to exist, but would not allow them to be elements of sets.

Briefly, von Neumann's system can be described as follows. As in Zermelo's theory, there is only one elementary predicate, namely the predicate $x \in Y$. The variables x, y, X, Y, etc. stand for objects which we agree to call *classes*; however, we distinguish between two kinds of classes, namely *elements*—which are defined to be those classes which are elements of classes—and *proper classes*, which are not elements of any class. Zermelo's axiom of selection is now replaced by a principle called the *class axiom*, which states the following:

> If $S(x)$ is any statement about an object x, there exists a class which consists of all those *elements x* which satisfy $S(x)$.

In other words, if $S(x)$ is any statement about x, we can form the class

$$\{x : x \text{ is an element and } S(x)\}.$$

To verify that Russel's paradox does not "work" in this system, let us go through the steps of Russell's argument, with S denoting $\{x : x \text{ is an element and } x \notin x\}$.

> $S \in S$ is impossible, for $S \in S$ implies $S \notin S$, which is a contradiction. Thus $S \notin S$. It follows that S is not an element, for, if S were an element, then we would have $S \in S$, which would be a contradiction.

Thus Russell's argument merely proves that the class S, defined above, is not an element.

The semantic paradoxes are avoided, as in the revised Zermelo system, by admitting in the class axiom only those "statements" which can be written in the formal language.

Variants of von Neumann's system have been developed by Gödel and Bernays. They have an advantage over Zermelo's system in that the class axiom is closer to the spirit of intuitive set theory than the axiom of selection. Indeed, if $S(x)$ is any statement about x, the class axiom guarantees the existence of a *class* containing all the elements x which satisfy $S(x)$. In mathematics, systems of the von Neumann type provide us with the convenience of being able to speak to the "class of all elements" and of being able to operate on classes which are not elements. (Such classes tend to occur at various points in higher mathematics; they can be avoided, but only at price.) The chief disadvantage of these systems is that the distinction between classes which are elements and classes which are not elements—a highly artificial one—must always be borne in mind; however, this disadvantage is undoubtedly outweighed by the greater flexibility and naturalness of von Neumann type systems. In this text we shall use a slightly modified form of von Neumann's system of axioms.

5 OBJECTIONS TO THE AXIOMATIC APPROACH. OTHER PROPOSALS

What are the chief goals of axiomatic set theory, and to what extent have these goals been successfully attained? In order to answer that question we must remember the circumstances which led mathematicians, in the early years of the twentieth century, to search for an axiomatic basis to set theory. The ideas of Cantor had already thoroughly permeated the fabric of modern mathematics, and had become indispensable tools of the working mathematician. Algebra and analysis were formulated within a framework of set theory, and some of the most elegant, powerful new results in these fields were established by using the methods introduced by Cantor and his followers. Thus, when the paradoxes were discovered and there arose doubts as to the basic validity of Cantor's system, most mathematicians were understandably reluctant to give it up; they trusted that some way would be found to circumvent the contradictions and preserve, if not all, at least most of Cantor's results. Hilbert once wrote in this connection: "We will not be expelled from the paradise into which Cantor has led us."

With the discovery of newer paradoxes, and the failure of all the initial attempts to avoid them, it became increasingly clear that it would not be possible to preserve intuitive set theory in its entirety. Something—possibly quite a lot—would have to be relinquished. The best that one could hope for was to retain as much of intuitive set theory as was needed to save the new results of modern mathematics and provide an adequate framework for classical mathematics.

Briefly, then, axiomatic set theory was created to achieve a limited aim: it had to provide a firm foundation for a system of set theory which—while it did not need to be as comprehensive as intuitive set theory—must include all of Cantor's basic results as well as the constructions (such as the number systems, functions, and relations) needed for classical mathematics.

The systems of both Zermelo and von Neumann were successful in achieving this limited aim. But the amount of intuitive set theory which they had to sacrifice was considerable. For example, in Zermelo's system, as we have already seen, the intuitive way of making sets—by naming a property of objects and forming the set of all objects which have that property—does not take place at all. It is replaced by the axiom of selection, in which properties are allowed only to determine *subsets of given sets*. Furthermore, the only admissible "properties" are those which can be expressed entirely in terms of the seven symbols ∈, ∨, ∧, ¬, ⇒, ∀, ∃ and variables *x, y, z*, … As a result, many of the things that we normally think of as sets—for example, the "set of all apples," the "set of all atoms in the universe"—are not admissible as "sets" in axiomatic set theory. In fact, the only "sets" which the axioms provide for are, first, the empty set Ø, and then constructions such as {Ø}, {Ø, {Ø}}, etc., which can be built up from the empty set. It is remarkable fact that all of mathematics can be based upon such a meager concept of set.

While the various axiomatic systems of set theory saved mathematics from its immediate peril, they failed to satisfy a great many people. In particular, many who were sensitive to the elegance and universality of mathematics were quick to point out that the creations of Zermelo and von Neumann must be regarded as provisional solutions—as expedients to solve a temporary problem; they will have to be replaced, sooner or later, by a mathematical theory of broader scope, which treats the concept of "set" in its full, intuitive generality.

This argument against axiomatic set theory—that it deals with an amputated version of our intuitive conception of a set—has important philosophical ramifications; it is part of a far wider debate, on the nature of mathematical "truth." The debate centers around the following question: Are mathematical concepts creations (that is, *inventions*) of the human mind, or do they exist independently of us in a "platonic" realm of concepts, merely to be *discovered* by the mathematician? The latter opinion is often referred to as "platonic realism" and is the dominant viewpoint of classical mathematics. We illustrate these two opposing points of view by showing how they apply to a particular concept—the notion of natural numbers. From the viewpoint of platonic realism, the concepts “one,” “two,” “three,” and so on, exist in nature and existed before the first man began to count. If intelligent beings exist elsewhere in the universe, then, no matter how different they are from us, they have no doubt discovered the natural numbers and found them to have the same properties they have for us. On the other hand, according to the opposing point of view, while three cows, three stones, or three trees exist in nature, the natural number *three* is a creation of our minds; we have invented a procedure for constructing the natural numbers (by starting from zero and adding 1 each time, thus producing successively 1, 2, 3, etc.) and have in this manner fashioned a conceptual instrument of our own making.

How does platonic realism affect the status of axiomatic set theory? From the point of view of platonic realism, mathematical objects are given to us ready-made, with all their features and all their properties. It follows that to say a mathematical theorem is *true* means it expresses a correct statement about the relevant mathematical objects. (For example, the proposition $2 + 2 = 4$ is not merely a formal statement provable in arithmetic; it states an *actual fact* about numbers.) Now—if we admit that mathematical objects are given to us with all their properties, it follows, in particular, that the notion of *set* is a fixed, well-defined concept which we are not free to alter for our own convenience. Thus the “sets” created by Zermelo and von Neumann do not exist, and theorems which purport to describe these nonexistent objects are false! In conclusion, if we were to accept a strict interpretation of platonic realism, we would be forced to reject the systems of Zermelo and von Neumann as mathematically invalid.

Fortunately, the trend, for some time now, has been away from platonism and toward a more flexible, more “agnostic” attitude toward mathematical “truth.” For one thing, developments in mathematics have been conforming less and less to the pattern dictated by platonic philosophy. For another, the cardinal requirement of platonism—that every mathematical object correspond to a definite, distinct object of our intuition (just as “point” and “line” refer to well-defined objects of our spatial intuition)—came to be an almost unbearable burden on the work of creative mathematicians by the nineteenth century. They were dealing with a host of new concepts (such as complex numbers, abstract laws of composition, and the general notion of function) which did not lend themselves to a simple interpretation in concrete terms. The case of the complex numbers is a good illustration of what was happening. Classical mathematics never felt at ease with the complex numbers, for it lacked a suitable “interpretation” of them, and as a result there were nagging doubts as to whether such things really “existed.” Real numbers may be interpreted as lengths or quantities, but the square root of a negative real number—this did not seem to correspond to anything in the real world or in our intuition of number. Yet the system of the complex numbers arises in a most natural way—as the smallest number system which contains the real numbers and includes the roots of every algebraic equation with real coefficients; whether or not the complex numbers have a physical or psychological counterpart seems irrelevant.

The case of the complex numbers strikes a parallel with the problem of axiomatic set theory. For the “sets” created by Zermelo and von Neumann arise quite naturally in a mathematical context. They give us the simplest notion of set which is adequate for mathematics and yields a consistent axiomatic theory. Whether or not we can interpret them intuitively may be relatively unimportant.

Be that as it may, many mathematicians in the early 1900’s were reluctant to make so sharp a break with tradition as axiomatic set theory seemed to demand. Furthermore, they felt, on esthetic grounds, that a mathematical theory of sets should describe all the things—and only those things—which our intuition recognizes to be sets. Among them was Bertrand Russell; in his efforts to reinstate intuitive set theory, Russell was led to the idea that we may consider sets to be ordered in a hierarchy of “levels,” where, if A and $\mathscr{B}$ are sets and A is an element of $\mathscr{B}$, then $\mathscr{B}$ is “one level higher” than A. For example, in plane geometry, a *circle* (regarded as a set of points) is one level below a *family of circles*, which, in turn, is one level below a *set of families of circles*. This basic idea was built by Russell into a theory called the *theory of types*, which can be described, in essence, as follows.

Every set has a natural number assigned to it, called its *level*. The simplest sets, those of level 0, are called *individuals*—they do not have elements. A collection of individuals is a set of level 1; a collection of sets of level 1 is a set of level 2; and so on. In the theory of types the expression $a \in B$ is only meaningful if, for some number n, a is a set of level n and B is a set of level $n + 1$. It follows that the statement $x \in x$ has no meaning in the theory of types, and as a result, Russell's paradox vanishes for the simple reason that it cannot even be formulated.

Russell's theory of types is built upon a beautifully simple idea. Unfortunately, in order to make it "work," Russell was forced to add a host of new assumptions, until finally the resulting theory became too cumbersome to work with and too complicated to be truly pleasing. For one thing, corresponding to the hierarchy of "levels" of sets, it was necessary to have a hierarchy of "level" of logical predicates. Then—as a way of avoiding the semantic paradoxes—sets at the same level were further divided into "orders." Finally, Russell had to admit a so-called *Axiom of Reducibility* which was just as arbitrary, just as ungrounded in intuition, as any of the *ad hoc* assumptions made by Zermelo. A a result of these shortcomings, the theory of types has not gained wide acceptance among mathematicians, although it is still an interesting (and maybe promising) area of research.

A far more radical approach was taken by a group of mathematicians calling themselves *intuitionists*. To the intuitionists, much of modern mathematics, including almost all of Cantor's theory of sets, is based on the uncritical use of rules of logic which they consider to be invalid. The intuitionist attitude toward set theory can therefore be summed up very easily: it is one of almost total rejection.

In order to properly understand the philosophy of intuitionism, we must first gain an understanding of its attitude toward logic. As the intuitionist sees it, the rules of logic used by mathematicians have an empirical character. Certain methods of proof came to be commonly used by mathematicians, and, over the years, were codified into a body of rules. These rules were observably correct in their original context, but—after they were codified—they came to be used uncritically in totally different contexts in which they no longer applied. Let us be more specific: in Euclidean geometry, which is the source of most mathematics before the fifteenth century, every theorem involves only a finite number of objects, and each of these objects (geometric figures) is given by an explicit construction. The rules of logic used by Euclid are perfectly valid in this context, say the intuitionists; it is only when they are transposed to problems involving an infinite domain of objects, or in which the objects are not given by an explicit construction, that the rules are incorrect.

As an example, let us take the *law of the excluded middle*. This is the rule which says that if S is any statement, then either S is true or the denial of S is true. In particular, let A be a set and let $P(x)$ be a statement which is meaningful for every element x in A. By the law of excluded middle, either *there exists an x in A such that $P(x)$ is true*, or else *for every x in A $P(x)$ is false*. Now the intuitionists will accept this rule if A is a finite set and if each of its elements can be tested to determine whether $P(x)$ is true or false. In fact, say the intuitionist, this is where the rule originated—our experience tells us that if we examine every element x in A to determine whether or not $P(x)$ is true (to do so, A has to be a finite set), there can be only two possible outcomes: either we have found an x for which $P(x)$ is true, or else, for every x in A we have found $P(x)$ to be false. Our experience, then, confirms the law of the excluded middle in the case of finite sets. But, say the intuitionists, to assume it is true in the case of infinite sets —in an area where we have no experience and experience is impossible—is wholly without foundation. On grounds such as these, the intuitionists deny the law of the excluded middle. Other rules of traditional logic are similarly rejected, because they go beyond the realm of our experience.

The intuitionist points out that mathematics originated as a study of certain mental constructions—chiefly geometric figures and simple constructions involving whole numbers. The theorems of early mathematics were essentially statements to the effect that if certain constructions are carried out, certain results will be achieved. For example, consider the following theorem of geometry: Given two triangles, if two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of the other triangle, then the two triangles are congruent. What is expressed here is the fact that if we construct two triangles, with two sides and an included angle equal as stated above, we will be able to verify (for example, by using a compass) that the remaining side of one triangle is equal to the remaining side of the other triangle. The proofs of Euclidean geometry, and of early mathematics generally, have a constructive character. For example, the Pythagorean theorem is proven by constructing a figure in which corresponding parts are congruent, hence have the same area. Once the construction is completed, it remains only to point out the congruent parts and thereby reach the desired conclusion. Thus, say the intuitionists, rules of logic were originally intended to describe situations which arose in the context of such constructions and determinations. For example, the rule of the excluded middle was intended simply to note the fact that if we are given a (finite) set of object and a method (for example, using a ruler and compass) to test each object for some property *P*, then, if we perform the test on every object, either one of the objects will pass the requirements of the test, or else every object will fail the requirements.

To sum up, then, the intuitionist maintains that a mathematical theorem is nothing more than a factual statement to the effect that a given mental construction will lead to a given result. Every proof must be constructive. If we claim that a mathematical object exists, we must prove it by giving a method for actually constructing the object. If we assert that a relation holds among given pairs of objects, our proof must include a method for testing every pair of objects in question. The “rules of logic” are nothing more than simple observations on the process of performing mathematical constructions; we have no grounds to believe these rules apply outside the context of constructive mathematics—in fact, it is meaningless to apply them outside this context. *Logic is incidental, not essential, to mathematics*.

It is obvious that the intuitionist’s notion of *set* must be quite different from ours. Consider, for example, Cantor’s principle that if we can name a property of objects, then there exists a set of all objects which have that property. Now this principle—as well as the limited versions of it accepted by Zermelo and von Neumann—is anathema to the intuitionists. An object exists only if it can be constructed; hence, a set exists only if we are able to describe a procedure for building it.

A full discussion of intuitionist set theory is outside the scope of this book; however, it is worth mentioning a particular kind of set which is important in intuitionist mathematics; this is called a *spread*. A spread is identified with a rule for producing all of its elements. Thus, a spread is not regarded as an “already formed” totality, but rather as a “process of formation”; each of its elements can be formed if we apply the rule long enough.

The paradoxes do not occur in intuitionist set theory because the crucial sets in the paradoxes cannot be produced in intuitionist mathematics, and the essential arguments cannot be rendered using intuitionist logic.

6 CONCLUDING REMARKS

During the early part of the twentieth century, as we have seen, various ways of building a noncontradictory theory of sets were proposed and developed by different “schools” of mathematicians. We have reviewed the basic principles of axiomatic set theory, Russell’s theory of types and the intuitionist (or “constructivist”) approach to sets. In addition to these, a great many other ideas were proposed, too various to describe in this brief introduction.

Of all the ways of dealing with sets, the axiomatic method seemed to best suit the needs of modern mathematics. The notion of “set” embodied in the system of Zermelo and von Neumann is broad enough for the purposes of mathematics, and therefore in a mathematical setting it is virtually indistinguishable from the Cantorian notion of set. The methods of proof, the symbolism, the rigor—all of these correspond to current mathematical usage, Most important of all, axiomatic set theory seems “natural” to most working mathematicians.

Those who reject axiomatic set theory do so on the basis of some philosophical bias. Those philosophical positions, however, which refuse to accept axiomatic set theory also deny the validity of a large part of modern mathematics. For example, the intuitionist school rejects the greater part of contemporary analysis, because it is founded on nonconstructivist principles. While the arguments of these critic present a challenge, and certainly give us food for thought, they are not powerful enough to destroy the achievements of three brilliant generations of mathematicians.

In the course of the past seventy years or so, set theory has come to be widely recognized as the fundamental, “unifying” branch of mathematics. We have already seen how the natural numbers can be constructed, and their properties derived, within the framework of set theory; from there, it is easy to develop the rational numbers, the real and complex numbers, as well as remarkable systems such as Cantor’s “transfinite cardinals.” The notions of function, relation, operation, and so forth are easily defined in terms of sets, and, as a result, every known branch of mathematics can be formulated within set theory. It is therefore legitimate—and, in fact, vital—to ask the question: “How secure a foundation does set theory provide for the whole edifice of mathematics?” In particular, are we absolutely certain that axiomatic set theory is consistent, that is, free of contradictions? If it is, then everything we develop within it—in other words, all of mathematics—is consistent; and if it is not, then whatever we build upon it is worthless.

The fact is that there is no known proof of the consistency of axiomatic set theory. This is not too surprising though, in view of some of the results of modern logic. For example, in 1931, K. Gödel proved that it is impossible to give a finitary proof of the consistency of ordinary arithmetic (of the natural numbers). The situation in almost every other branch of mathematics is much the same. Thus the best assurance that we have, at this time, of the consistency of axiomatic set theory is the fact that the familiar contradictions cannot be obtained in the usual way. We cannot do any better at this time.

A result relating to *relative consistency* is of some interest. It has been proven recently that *if* Zermelo’s axiomatization of set theory is consistent, *then* von Neumann’s axiomatization is also consistent.

It is probable that, in the final analysis, any assurance of the consistency of mathematics will have to rest on some combination of basic intuition and empirical evidence.

* The four rules given in (3) above, together with seven additional rules, make up the system of natural deduction described in Slupecki and Borkowski [8]; these eleven rules are sufficient for every valid logical argument. Other systems are described in Quine [6] and Suppes [9].

1
Classes and Sets

1 BUILDING SENTENCES

Before introducing the basic notions of set theory, it will be useful to make certain observations on the use of language.

By a *sentence* we will mean a statement which, in a given context, is unambiguously either *true* or *false*. Thus

London is the capital of England.

Money grows on trees.

Snow is black.

are examples of sentences. We will use letters P, Q, R, S, etc., to denote sentences; used in this sense, P, for instance, is to be understood as asserting that "P is true."

Sentences may be combined in various ways to form more complicated sentences. Often, the truth or falsity of the compound sentence is completely determined by the truth or falsity of its component parts. Thus, if P is a sentence, one of the simplest sentences we may form from P is the **negation** of P, denoted by $\neg P$ (to be read "not P "), which is understood to assert that "P is *false*." Now if P is true, then, quite clearly, $\neg P$ is false; and if P is false, then $\neg P$ is true. It is convenient to display the relationship between $\neg P$ and P in the following *truth table*,

1.1

P	$\neg P$
t	f
f	t

where t and f denote the "truth values", *true* and *false*.

Another simple operation on sentences is conjunction: if P and Q are sentences, the **conjunction** of P and Q, denoted by $P \wedge Q$ (to be read "P and Q"), is understood to assert that "*P is true and Q is true*." It is intuitively clear that $P \wedge Q$ is true if P and Q are both true, and false otherwise; thus, we have the following truth table.

1.2

P	Q	$P \wedge Q$
t	t	t
t	f	f
f	t	f
f	f	f

The **disjunction** of P and Q, denoted by $P \vee Q$ (to be read "P or Q"), is the sentence which asserts that “*P is true, or Q is true, or P and Q are both true*.” It is clear that $P \vee Q$ is false only if P and Q are both false.

1.3

P	Q	$P \vee Q$
t	t	t
t	f	t
f	t	t
f	f	f

An especially important operation on sentences is **implication** : if P and Q are sentences, then $P \Rightarrow Q$ (to be read “P implies Q”) asserts that “*if P is true, then Q is true*.” *A word of caution*: in ordinary usage, “if P is true, then Q is true” is understood to mean that there is a *causal* relationship between P and Q (as in “if John passes the course, then John can graduate”). In mathematics, however, implication is always understood in the *formal* sense: $P \Rightarrow Q$ is true *except if* P is true and Q is false. In other words, $P \Rightarrow Q$ is defined by the truth table.

1.4

P	Q	$P \Rightarrow Q$
t	t	t
t	f	f
f	t	t
f	f	t

The properties of formal implication differ somewhat from the properties we would expect “causal” implication to have. For example,

$$\text{“}1+1=2 \quad \Rightarrow \quad \pi \text{ is a transcendental number”}$$

is true, even though there is no causal relationship between the two component sentences. To take another example,

$$\text{“}2+2=5 \quad \Rightarrow \quad 4 \text{ is a prime number”}$$

is true, even though the two component sentences are false. This should not disturb the reader unduly, for formal implication still has the fundamental property which we demand of implication—namely, if $P \Rightarrow Q$ is true, then, necessarily, if P is true then Q is true.

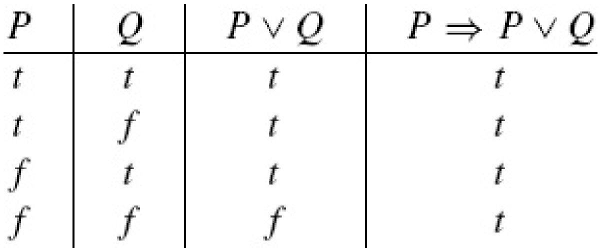
Certain compound sentences are true regardless of the truth or falsity of their component parts; a typical example is the sentence $P \Rightarrow P$. Regardless of whether P is true or false, $P \Rightarrow P$ is always true; in other words, no matter what sentence P is, $P \Rightarrow P$ is true. For future reference we record a few sentences which have this property.

1.5 Theorem For all sentences P and Q, the following statements are true.

i) $P \Rightarrow P \vee Q$. i)′ $Q \Rightarrow P \vee Q$.
ii) $P \wedge Q \Rightarrow P$. ii)′ $P \wedge Q \Rightarrow Q$.

Proof

i) We wish to prove that if P and Q are any sentences, then $P \Rightarrow P \vee Q$ is true; in other words, we wish to prove that no matter what truth values are assumed by P and Q, $P \Rightarrow P \vee Q$ is always true. To do this, we derive a truth table for $P \Rightarrow P \vee Q$ as follows.

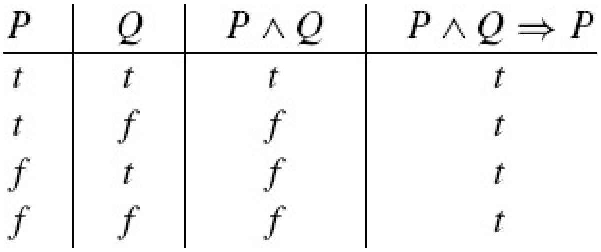


P	Q	$P \vee Q$	$P \Rightarrow P \vee Q$
t	t	t	t
t	f	t	t
f	t	t	t
f	f	f	t

The basic idea of the derived truth table is this: in line 1, P and Q both take the value t; thus, by 1.3, $P \vee Q$ takes the value t;now, P has the value t and $P \vee Q$ has the value t, so, by 1.4, $P \Rightarrow P \vee Q$ takes the value t. We do the same for each line, and we find that in every line (that is, for every possible assignment of truth values to P and Q) $P \Rightarrow P \vee Q$ has the value t (true). This is what we had set out to prove.

i)′ The derived truth table for $Q \Rightarrow P \vee Q$ is analogous to the one for $P \Rightarrow P \vee Q$; the conclusion is the same.

ii) In order to prove that $P \wedge Q \Rightarrow P$ for all sentences P and Q, we derive a truth table for $P \wedge Q \Rightarrow P$.



P	Q	$P \wedge Q$	$P \wedge Q \Rightarrow P$
t	t	t	t
t	f	f	t
f	t	f	t
f	f	f	t

In every line (that is, for every possible assignment of truth values to P and Q), $P \wedge Q \Rightarrow P$ takes the value t; thus, $P \wedge Q \Rightarrow P$ is true irrespective of the truth or falsity of its component sentences P and Q.

ii)′ The truth table for $P \wedge Q \Rightarrow Q$ is analogous to the one for $P \wedge Q \Rightarrow P$, and the conclusion is the same. ■

1.6 Theorem For all sentences P, Q and R, the following is true:

$$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R).$$

Proof. The reader should derive the truth table for

$$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$$

and verify that this sentence takes the truth value t in every line of the table. ■

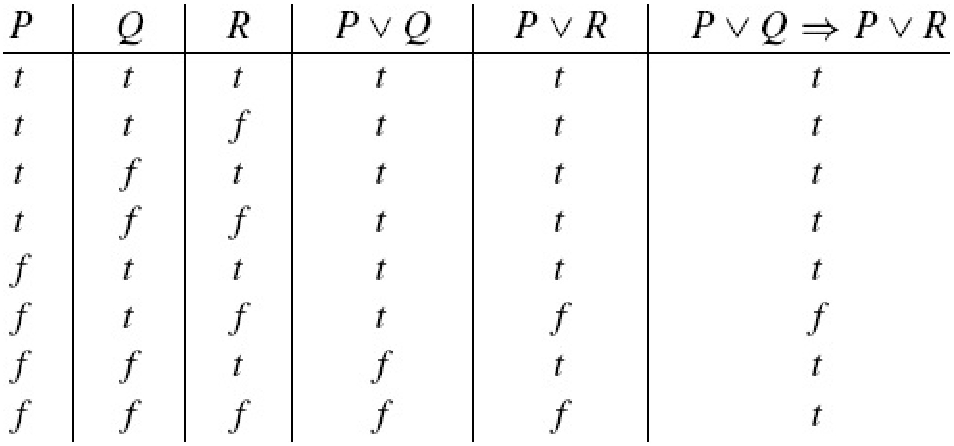
1.7 Theorem For all sentences P, Q and R, if $Q \Rightarrow R$ is true, then

i) $P \lor Q \Rightarrow P \lor R$ is true, and

ii) $P \land Q \Rightarrow P \land R$ is true.

Proof

i) We assume that $Q \Rightarrow R$ is true, and derive the truth table for $P \lor Q \Rightarrow P \lor R$.



P	Q	R	$P \lor Q$	$P \lor R$	$P \lor Q \Rightarrow P \lor R$
t	t	t	t	t	t
t	t	f	t	t	t
t	f	t	t	t	t
t	f	f	t	t	t
f	t	t	t	t	t
f	t	f	t	f	f
f	f	t	f	t	t
f	f	f	f	f	t

Since we assume that $Q \Rightarrow R$ is true, we cannot have, simultaneously, Q true and R false; thus, we may disregard the sixth line of the table. In all of the remaining lines, $P \lor Q \Rightarrow P \lor R$ takes the value t.

ii) The proof that $P \land Q \Rightarrow P \land R$ is analogous to the above. ■

We agree that $P \Leftrightarrow Q$ is to be an abbreviation for $(P \Rightarrow Q) \land (Q \Rightarrow P)$.

1.8 Theorem For all sentences P, Q and R, the following are true:

i) $P \lor Q \Leftrightarrow Q \lor P$, i)′ $P \land Q \Leftrightarrow Q \land P$,

ii) $P \lor (Q \lor R) \Leftrightarrow (P \lor Q) \lor R$, ii)′ $P \land (Q \land R) \Leftrightarrow (P \land Q) \land R$,

iii) $P \land (Q \lor R) \Leftrightarrow (P \land Q) \lor (P \land R)$, iii)′ $P \lor (Q \land R) \Leftrightarrow (P \lor Q) \land (P \lor R)$,

iv) $P \lor P \Leftrightarrow P$, iv)′ $P \land P \Leftrightarrow P$.

The proof of this theorem is left as an exercise for the reader.

In this and the subsequent chapters, ⇒ will be used as an abbreviation for *implies*, ⇔ will be used as an abbreviation for *if and only if* (we will sometimes write “iff” instead of ⇔), ∧ will be used as an abbreviation for *and*, and ∨ will be used as an abbreviation for *or*. If $P, Q, R, \ldots$ are any statements, an expression of the form $P \Rightarrow Q \Rightarrow R \Rightarrow \ldots$ should be understood to mean that $P \Rightarrow Q$, $Q \Rightarrow R$, and so on; analogously, $P \Leftrightarrow Q \Leftrightarrow R \Leftrightarrow \ldots$ should be understood to mean that $P \Leftrightarrow Q$, $Q \Leftrightarrow R$, and so on.

As is customary, ∃ is to be read *there exists*, ∀ is to be read *for all*, and ∋ is to be read *such that*.

EXERCISES 1.1

1. Prove Theorem 1.8.

2. Prove that the following sentences are true for all P and Q (DeMorgan’s Laws).

a) $\neg(P \lor Q) \Leftrightarrow \neg P \land \neg Q$. b) $\neg(P \land Q) \Leftrightarrow \neg P \lor \neg Q$.

3. Prove that the following sentences are true, for every sentence *P*.
   a) $\neg\neg P \Rightarrow P$. b) $P \Rightarrow \neg\neg P$.

4. Prove that the following sentences are true for all *P* and *Q*.
   a) $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$. b) $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.
   c) $(P \Rightarrow Q) \Leftrightarrow \neg(P \wedge \neg Q)$. d) $[P \wedge (P \Rightarrow Q)] \Rightarrow Q$.
   e) $[(P \vee Q) \wedge \neg P] \Rightarrow Q$.

5. Prove the following sentences are true for all *P*, *Q* and *R*.
   a) $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$.
   b) $[(P \Rightarrow Q) \wedge (R \Rightarrow Q)] \Leftrightarrow [(P \vee R) \Rightarrow Q]$.
   c) $[(P \Rightarrow Q) \wedge (P \Rightarrow R)] \Leftrightarrow [P \Rightarrow (Q \wedge R)]$.

6. Prove that, for all sentences *P*, *Q* and *R*, if $Q \Leftrightarrow R$ is true, then the following are true.
   a) $P \vee Q \Leftrightarrow P \vee R$. b) $P \wedge Q \Leftrightarrow P \wedge R$.
   c) $(P \Rightarrow Q) \Leftrightarrow (P \Rightarrow R)$.

7. Prove that for all sentences *P*, *Q*, *R* and *S*, if $P \Rightarrow Q$ and $R \Rightarrow S$, then
   a) $P \vee R \Rightarrow Q \vee S$. b) $P \wedge R \Rightarrow Q \wedge S$.

2 BUILDING CLASSES

We will now begin our development of axiomatic set theory.

Every axiomatic system, as we have seen, must start with a certain number of *undefined notions*. For example, in geometry, the words “point” and “line” are generally taken to be undefined. While we are free in our own minds to attach a “meaning,” in the form of a mental picture, to each of these notions, mathematically we must proceed “as if” we did not know what they meant. Now an “undefined” notion has no properties except those which are explicitly assigned to it; therefore, we must state as *axioms* all the elementary properties which we expect our undefined notions to have.

Our system of axiomatic set theory is based on just two undefined notions: The word *class* and the *membership relation* ∈. All the objects of our theory are called classes. It was explained in Chapter 1 that in order to avoid logical paradoxes, we have to distinguish between two kinds of classes—those that are called *sets* and those that are called *proper classes*. We shall say no more about this distinction now, but shall return to it later.

If *x* and *A* are classes, the expression $x \in A$ is read “*x* is an element of *A*”, or “*x* belongs to *A*”, or simply “*x* is in *A*”. It is convenient to write $x \notin A$ for “*x* is not an element of *A*“.

Definition Let *x* be a class. If *x* is an element of some class *A* then *x* is called an **element**.

We shall have more to say about elements and classes in the final section of this chapter. From here on, we shall use the following notational convention: **lower-case letters** *a*, *b*, *c*, *x*, *y*, … **will be used only to designate elements**. Thus, a capital letter, such as *A*, may denote either an element or a class which is not an element, but a lower-case letter, such as *x*, may denote *only* an element. Intuitively, two classes should be called equal if they are elements of the same classes.

1.9 Definition Let *A* and *B* be classes. We define *A* = *B* to mean that every class that has *A* as an element also has *B* as an element, and vice-versa. In symbols,

$$\mathrm{A} = \mathrm{B} \quad \text{iff} \quad (\forall X)[A \in X \Rightarrow B \in X \text{ and } B \in X \Rightarrow A \in X]$$

We have defined two classes to be equal if and only if they are members of the same classes. Informally, we may therefore think of them as interchangeable. Equal classes have another property. If A and B are equal, we expect them to have the same elements. This property is stated as our first axiom:

A1. A = B iff $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.

This axiom is sometimes called the *Axiom of Extent*.

1.10 Definition Let A and B be classes; we define $A \subseteq B$ to mean that every element of A is an element of B. In symbols,

$$A \subseteq B \quad \text{iff} \quad x \in A \Rightarrow x \in B.$$

If $A \subseteq B$, then we say that A is a *subclass* of B.

We define $A \subset B$ to mean that $A \subseteq B$ and $A \neq B$; in this case, we say that A is a *strict subclass* of B.

If A is a subclass of B, and A is a set, we will call A a *subset* of B.

A few simple properties of equality and inclusion are given in the next theorem.

1.11 Theorem For all classes A, B and C, the following hold:

i) $A = A$.

ii) $A = B \Rightarrow B = A$.

iii) $A = B$ and $B = C \Rightarrow A = C$.

iv) $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$.

v) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

Proof

i) The statement $x \in A \Rightarrow x \in A$ and $x \in A \Rightarrow x \in A$ is obviously true; thus, by Axiom A1, $A = A$.

ii) Suppose $A = B$; then $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$; hence by 1.8(i)′$x \in B \Rightarrow x \in A$ and $x \in A \Rightarrow x \in B$; thus, by Axiom A1, $B = A$.

iii) Suppose $A = B$ and $B = C$; then we have the following:

$$\begin{aligned} x \in A &\Rightarrow x \in B, \\ x \in B &\Rightarrow x \in A, \\ x \in B &\Rightarrow x \in C, \\ x \in C &\Rightarrow x \in B. \end{aligned}$$

From the first and third of these statements we conclude (by 1.6) that $x \in A \Rightarrow x \in C$. From the second and fourth of these statements we conclude that $x \in C \Rightarrow x \in A$. Thus, by Axiom A1, $A = C$.

We leave the proofs of (iv) and (v) as an exercise for the reader. ■

We have seen that the intuitive way of making classes is to name a property of objects and form the class of all the objects which have that property. Our second axiom allows us to make classes in this manner.

A2. Let *P(x)* designate a statement about *x* which can be expressed entirely in terms of the symbols ∈, ∨, ∧, ¬, ⇒, ∃, ∀, brackets, and variables *x, y, z, A, B, …* Then there exists a class *C* which consists of all the elements *x* which satisfy *P(x)*.

Axiom A2 is called the *axiom of class construction*.

The reader should note that axiom A2 permits us to form the class of all the *elements x* which satisfy *P(x)*, *not* the class of all the *classes x* which satisfy *P(x)*;as discussed on page 13, this distinction is sufficient to eliminate the logical paradoxes.
The semantic paradoxes have been avoided by admitting in axiom A2 only those statements *P(x)* which can be written entirely in terms of the symbols ∈, ∨, ∧, ¬, ⇒, ∃, ∀, brackets and variables.
The class *C* whose existence is asserted by Axiom A2 will be designated by the symbol

$$C = \{x : P(x)\}.$$

1.12 *Remark.* The use of a small *x* in the expression {*x* : *P(x)*} is not accidental, but quite essential. Indeed, we have agreed that lower-case letters *x, y*, etc., will be used *only* to designate elements. Thus

$$C = \{x : P(x)\}$$

asserts that *C* is the class of all the *elements x* which satisfy *P(x)*.

We will now use the axiom of class construction to build some new classes from given classes.

1.13 Definition Let *A* and *B* be classes; the *union* of *A* and *B* is defined to be the class of all the elements which belong either to *A*,or *B*, or to both *A* and *B*. In symbols,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Thus, $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

1.14 Definition Let *A* and *B* be classes; the *intersection* of *A* and *B* is defined to be the class of all the elements which belong to both *A* and *B*. In symbols,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Thus, $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

1.15 Definition By the *universal class* $\mathcal{U}$ we mean the class of all elements. The existence of the universal class is a consequence of the axiom of class construction, for if we take *P(x)* to be the statement *x* = *x*, then A2 guarantees the existence of a class which consists of all the elements which satisfy *x* = *x*; by 1.11(i), every element is in this class.

1.16 Definition By the *empty class* we mean the class Ø which has no elements at all. The existence of the empty class is a consequence of the axiom of class construction; indeed, A2 guarantees the existence of a class which consists of all the elements which satisfy $x \neq x$; by Theorem 1.11(i), this class has no elements.

1.17 Theorem For every class *A*, the following hold:

i) $\emptyset \subseteq A$. ii) $A \subseteq \mathcal{U}$.

Proof

i) In order to prove that $\emptyset \subseteq A$, we must show that $x \in \emptyset \Rightarrow x \in A$. It suffices to prove the contrapositive of this statement, that is, $x \notin A \Rightarrow x \notin \emptyset$. Well, suppose $x \notin A$; then certainly $x \notin \emptyset$, for Ø has no elements; thus $x \notin A \Rightarrow x \notin \emptyset$.

ii) If $x \in A$, then x is an element; hence $x \in \mathcal{U}$. ■

1.18 Definition If two classes have no elements in common, they are said to be *disjoint*. In symbols,

$$A \text{ and } B \text{ are disjoint} \quad \text{iff} \quad A \cap B = \emptyset.$$

1.19 Definition The *complement* of a class *A* is the class of all the elements which do not belong to *A*. In symbols,

$$A' = \{x : x \notin A\}.$$

Thus, $x \in A'$ if and only if $x \notin A$.

Relations among classes can be represented graphically by means of a useful device known as the *Venn diagram*. A class is represented by a simple plane area (circular or oval in shape); if it is desired to show the complement of a class, then the circle or oval is drawn within a rectangle which represents the universal class. Thus, $A \cup B$ is rendered by the shaded area of Fig. 1, $A \cap B$ by the shaded area of Fig. 2, and A' by the shaded area of Fig. 3. The reader will find that Venn diagrams are helpful in guiding his reasoning about classes, and that they give more meaning to set-theoretic formulas by making them more concrete. For example, in Section 3 of this chapter we will prove the formula

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

This formula is illustrated in Fig. 4, where the shaded area represents $A \cap (B \cup C)$; one immediately notices that this same shaded area represents $(A \cap B) \cup (A \cap C)$.

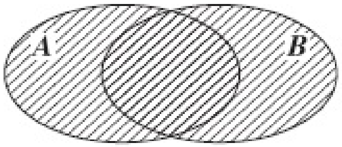


Fig.1

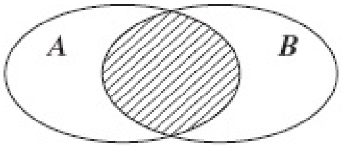


Fig.2

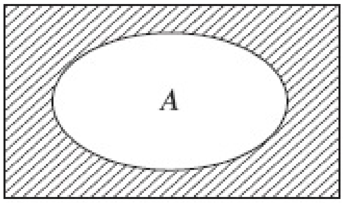


Fig.3

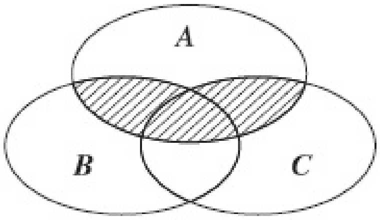


Fig.4

EXERCISES 1.2

1. Suppose that $A \subseteq B$ and $C \subseteq D$; prove that
   a) $(A \cup C) \subseteq (B \cup D)$, b) $(A \cap C) \subseteq (B \cap D)$.
   [*Hint:* Use the result of Exercise 7, Exercise Set 1.1.]
2. Suppose $A = B$ and $C = D$; prove that
   a) $A \cup C = B \cup D$, b) $A \cap C = B \cap D$.
   [*Hint*: Use the result of the preceding exercise.]
3. Prove that if $A \subseteq B$, then $B' \subseteq A'$.
   [*Hint*: Use the result of Exercise 4(a), Exercise Set 1.1.]
4. Prove that if $A = B$, then $A' = B'$.
5. Prove that if $A = B$ and $B \subseteq C$, then $A \subseteq C$.
6. Prove that if $A \subset B$ and $B \subset C$, then $A \subset C$.
7. Prove Theorem 1.11, parts (iv) and (v).
8. Let $S = \{x : x \notin x\}$; use Russell’s argument to prove that S is not an element.
9. Does Axiom A2 allow us to form the “class of all classes”? Explain.
10. Explain why Russell’s paradox and Berry’s paradox cannot be produced by using Axiom A2.

3 THE ALGEBRA OF CLASSES

One of the most interesting and useful facts about classes is that under the operations of union, intersection, and complementation they satisfy certain algebraic laws from which we can develop an algebra of classes. We shall see later (Chapter 4) that the algebra of classes is merely one example of a structure known as a *Boolean algebra*; another example is the "algebra of logic," where $\vee$, $\wedge$, $\neg$ are regarded as operations on sentences.

1.20 Theorem If A and B are any classes, then

i) $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

ii) $A \cap B \subseteq A$ and $B \cap B \subseteq B$.

Proof

i) To prove that $A \subseteq A \cup B$, we must show that $x \in A \Rightarrow x \in A \cup B$:

$$\begin{aligned} x \in A &\Rightarrow x \in A \;\vee\; x \in B && \text{by 1.5(i)} \\ &\Rightarrow x \in A \cup B && \text{by 1.13.} \end{aligned}$$

Analogously, we can show that $B \subseteq A \cup B$.

ii) To prove that $A \cap B \subseteq A$, we must show that $x \in A \cap B \Rightarrow x \in A$.

$$\begin{aligned} x \in A \cap B &\Rightarrow x \in A \;\wedge\; x \in B && \text{by 1.14} \\ &\Rightarrow x \in A \;\blacksquare && \text{by 1.5(ii).} \end{aligned}$$

1.21 Theorem If A and B are classes, then

i) $A \subseteq B$ if and only if $A \cup B = B$,

ii) $A \subseteq B$ if and only if $A \cap B = A$.

Proof

i) Let us first assume that $A \subseteq B$; that is, $x \in A \Rightarrow x \in B$. Then

$$\begin{aligned} x \in A \cup B &\Rightarrow x \in A \quad \text{or} \quad x \in B && \text{by 1.13} \\ &\Rightarrow x \in B \quad \text{or} \quad x \in B && \text{by 1.7(i)} \\ &\Rightarrow x \in B && \text{by 1.8(iv).} \end{aligned}$$

Thus, $A \cup B \subseteq B$;but $B \subseteq A \cup B$ by 1.20(i); consequently, $A \cup B = B$.

Conversely, let us assume that $A \cup B = B$. By 1.20(i), $A \subseteq A \cup B$; thus $A \subseteq B$.

ii) To proof is left as an exercise for the reader. ■

1.22 Theorem (*Absorption Laws*). For all classes A and B,

i) $A \cup (A \cap B) = A$. ii) $A \cap (A \cup B) = A$.

Proof

i) By 1.20(ii), $A \cap B \subseteq A$; therefore, by 1.21(i), $A \cup (A \cap B) = A$.

ii) By 1.20(i), $A \subseteq A \cup B$; therefore, by 1.21(ii), $A \cap (A \cup B) = A$. ■

1.23 Theorem For every class A, $(A')' = A$.

Proof

$$x \in (A')' \Rightarrow x \notin A' \Rightarrow x \in A \qquad \text{by 1.19,}$$

$$x \in (A) \Rightarrow x \notin A' \Rightarrow x \in (A')' \qquad \text{by 1.19. ■}$$

1.24 Theorem (*DeMorgan's Laws*). For all classes A and B,

i) $(A \cup B)' = A' \cap B'$. ii) $(A \cap B)' = A' \cup B'$.

Proof

i) First, $x \in (A \cup B)' \Rightarrow x \notin A \cup B$ by 1.19

$\Rightarrow x \notin A$ and $x \notin B$

(because if either $x \in A$ or $x \in B$, then $x \in A \cup B$)

$\Rightarrow x \in A'$ and $x \in B'$ by 1.19

$\Rightarrow x \in (A' \cap B')$ by 1.14.

Next, $x \in (A' \cap B') \Rightarrow x \in A'$ and $x \in B'$ by 1.14

$\Rightarrow x \notin A$ and $x \notin B$ by 1.19

$\Rightarrow x \notin A \cup B$

$\Rightarrow x \in (A \cup B)'$ by 1.19.

ii) The proof is left as an exercise for the reader. ■

1.25 Theorem For all classes A, B and C, the following are true.

Commutative Laws: i) $A \cup B = B \cup A$
ii) $A \cap B = B \cap A$

Idempotent Laws: iii) $A \cup A = A$
iv) $A \cap A = A$

Associative Laws: v) $A \cup (B \cup C) = (A \cup B) \cup C$
vi) $A \cap (B \cap C) = (A \cap B) \cap C$

Distributive Laws: vii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
viii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof

$$\begin{array}{lll} \text{i)}\ x \in A \cup B & \Rightarrow x \in A \quad \text{or} \quad x \in B & \text{by 1.13} \\ & \Rightarrow x \in B \quad \text{or} \quad x \in A & \text{by 1.8(i)} \\ & \Rightarrow x \in B \cup A & \text{by 1.13.} \end{array}$$

$$\begin{array}{lll} \text{v)}\ x \in A \cup (B \cup C) & \Rightarrow x \in A \ \vee \ x \in B \cup C & \text{by 1.13} \\ & \Rightarrow x \in A \ \vee \ (x \in B \ \vee \ x \in C) & \text{by 1.13} \\ & \Rightarrow (x \in A \ \vee \ x \in B) \ \vee \ x \in C & \text{by 1.8(ii)} \\ & \Rightarrow x \in A \cup B \ \vee \ x \in C & \text{by 1.13} \\ & \Rightarrow x \in (A \cup B) \cup C & \text{by 1.13.} \end{array}$$

$$\begin{array}{lll} \text{vii)}\ x \in A \cap (B \cup C) & \Rightarrow x \in A \ \wedge \ x \in B \cup C & \text{by 1.14} \\ & \Rightarrow x \in A \ \wedge \ (x \in B \vee x \in C) & \text{by 1.13} \\ & \Rightarrow (x \in A \ \wedge \ x \in B) \vee (x \in A \wedge x \in C) & \text{by 1.8(iii)} \\ & \Rightarrow x \in A \cap B \ \vee \ x \in A \cap C & \text{by 1.14} \\ & \Rightarrow x \in (A \cap B) \cup (A \cap C) & \text{by 1.13.} \end{array}$$

The proofs of (ii), (iii), (iv), (vi), and (viii) are exercises for the reader. ■

The empty class and the universal class are identity elements for union and intersection respectively; they satisfy the following simple rules:

1.26 Theorem For every class *A*,

$$\begin{array}{rlrl} \text{i)} & A \cup \varnothing = A. & \text{ii)} & A \cap \varnothing = \varnothing. \\ \text{iii)} & A \cup \mathcal{U} = \mathcal{U}. & \text{iv)} & A \cap \mathcal{U} = A. \\ \text{v)} & \mathcal{U} = \varnothing. & \text{vi)} & \varnothing' = \mathcal{U}. \\ \text{vii)} & A \cup A' = \mathcal{U}. & \text{viii)} & A \cap A' = \varnothing. \end{array}$$

Proof

i) By 1.17, $\varnothing \subseteq A$, and therefore by 1.21(i), $A \cup \varnothing = A$.

iii) By 1.17(ii), $A \subseteq \mathcal{U}$, and therefore by 1.21(i), $A \cup \mathcal{U} = A$.

The proofs of the remaining parts of this theorem are left as an exercise for the reader. ■

By using the laws of class algebra which we have developed above, we can prove all the elementary properties of classes without referring to the definitions of the symbols ∪, ∩, ′, and ⊆. The following is an example of how such proofs are carried out.

Example Prove that $A \cap (A' \cup B) = A \cap B$.

Proof

$$\begin{aligned} A \cap (A' \cup B) &= (A \cap A') \cup (A \cap B) && \text{by 1.25(vii)} \\ &= \emptyset \cup (A \cap B) && \text{by 1.26(viii)} \\ &= A \cap B && \text{by 1.26(i).} \end{aligned}$$

The following definition is frequently useful: The *difference* of two classes *A* and *B* is the class of all elements which belong to *A*, but do not belong to *B*. In symbols,

$$A - B = A \cap B'.$$

Example Prove that $A - B = B' - A'$.

Proof

$$\begin{aligned} A - B &= A \cap B' && \text{Definition} \\ &= B' \cap A && \text{by 1.25(ii)} \\ &= B' \cap (A')' && \text{by 1.23} \\ &= B' - A' && \text{Definition of } B' - A'. \end{aligned}$$

It is useful to note that with the aid of Theorem 1.21, relations involving inclusion $(\subseteq)$, not merely equality, can be proved using class algebra.

EXERCISES 1.3

1. Prove Theorem 1.21(ii).
2. Prove Theorem 1.24(ii).
3. Prove Theorem 1.25, parts (ii), (iii), (iv), (vi) and (viii).
4. Prove Theorem 1.26, parts (ii), (iv), (v) through (viii).
5. Use class algebra to prove the following.
   a) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$, b) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
6. Use class algebra to prove the following.
   a) If $A \cap C = \emptyset$, then $A \cap (B \cup C) = A \cap B$.
   b) If $A \cap B = \emptyset$, then $A - B = A$.
   c) If $A \cap B = \emptyset$ and $A \cup B = C$, then $A = C - B$.
7. Using class algebra, prove each of the following.
   a) $A \cap (B - C) = (A \cap B) - C$.
   b) $(A \cup B) - C = (A - C) \cup (B - C)$.
   c) $A - (B \cup C) = (A - B) \cap (A - C)$.
   d) $A - (B \cap C) = (A - B) \cup (A - C)$.
8. We define the operation + on classes as follows: If *A* and *B* are classes, then

$$A + B = (A - B) \cup (B - A).$$

   Prove each of the following.

a) $A+B=B+A$, b) $A+(B+C)=(A+B)+C$,
c) $A\cap(B+C)=(A\cap B)+(A\cap C)$, d) $A+A=\varnothing$, e) $A+\varnothing=A$.

9. Prove each of the following.
   a) $A \cup B = \varnothing \Rightarrow A = \varnothing$ and $B = \varnothing$.
   b) $A \cap B' = \varnothing$ if and only if $A \subseteq B$.
   c) $A + B = \varnothing$ if and only if $A = B$.
10. Prove each of the following.
    a) $A \cup C = B \cup C$ if and only if $A + B \subseteq C$.
    b) $(A \cup C) + (B \cup C) = (A + B) - C$.
11. Use class algebra to prove that $A \subseteq B$ and $C = B - A$, then $A = B - C$.

4 ORDERED PAIRS CARTESIAN PRODUCTS

If *a* is an element, we may use the axiom of class construction to form the class

$$\{a\} = \{x : x = a\}.$$

It is easy to see that $\{a\}$ contains only one element, namely the element *a*. A class containing a single element is called a *singleton*.

If *a* and *b* are elements, we may use the axiom of class construction to form the class

$$\{a, b\} = \{x : x = a \quad \text{or} \quad x = b\}.$$

Clearly $\{a, b\}$ contains two elements, namely the elements *a* and *b*. A class containing exactly two elements is called an *unordered pair*, or, more simply, a *doubleton*.

In like fashion, we can form the classes $\{a, b, c\}$, $\{a, b, c, d\}$, and so on.

1.27 Theorem If $\{x, y\}=\{u, v\}$, then

$$[x = u \quad \text{and} \quad y = u] \quad \text{or} \quad [x = v \quad \text{and} \quad y = u].$$

Proof

Suppose $\{x, y\}=\{u, v\}$; we will consider two cases, according as $x = y$ or $x \neq y$.

Case 1: $x = y$. Now $u \in \{u, v\}$ and $\{u, v\}=\{x, y\}$, so by Axiom A1, $u \in \{x, y\}$.

Thus, $u = x$ or $u = y$; in either case, $u = x = y$. Analogously, $v = x = y$,so we have $u = v = x = y$, and we are done.

Case 2: $x \neq y$. Now $x \in \{x, y\}$ and $\{x, y\}=\{u, v\}$,so $x \in \{u, v\}$; thus $x = u$ or $x = v$. We will consider these two cases separately:

i) $x = u$: Now $y \in \{x, y\}$, hence $y \in \{u, v\}$,so $y = u$ or $y = v$; but $x = u$,so if $y = u$, then $x = y$, which is impossible because we assume $x =y$. Thus $y = v$. In this case, we are done.
ii) $x = v$: We repeat the argument of (i), merely switching the roles of *u* and *v*;we get $y = u$; hence, again, we are done. ■

An important notion in mathematics is that of an *ordered pair* of elements. Intuitively, an ordered pair is a class consisting of two elements in a specified order. In fact, the order is not really essential; what is essential is that ordered pairs have the following property.

1.28 Let *(a, b)* and *(c, d)* be ordered pairs. If *(a, b)* = *(c, d)*, then *a* = *c* and *b* = *d*.

We would like to define ordered pairs in such a way as to avoid introducing a new undefined notion of "order." It is an interesting fact that this can, indeed, be accomplished; we proceed as follows.

1.29 Definition Let *a* and *b* be elements; the *ordered pair (a, b)* is defined to be the class

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

It is worth noting that

$$(b, a) = \{\{b\}, \{b, a\}\} = \{\{b\}, \{a, b\}\}.$$

Hence there is a clear distinction between the two possible "orders" *(a, b)* and *(b, a)*: they are different classes. It remains to prove that ordered pairs, as we have just defined them, have Property 1.28.

1.30 Theorem If *(a, b)* = *(c, d)*, then *a* = *c* and *b* = *d*.

Proof. Suppose that *(a, b)* = *(c, d)*; that is,

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}.$$

By Theorem 1.27, either

$$[\{a\} = \{c\} \quad \text{and} \quad \{a, b\} = \{c, d\}],$$

or

$$[\{a\} = \{c, d\} \quad \text{and} \quad \{a, b\} = \{c\}];$$

we will consider these two cases separately.

Case 1: $\{a\}=\{c\}$ and $\{a, b\}=\{c, d\}$. From $\{a\}=\{c\}$, it follows that $a = c$. From $\{a, b\}=\{c, d\}$ and Theorem 1.27, it follows that either $a = c$ and $b = d$,or $a = d$ and $b = c$; in the first case, we are done; in the second case, we have $b = c = a = d$, so again we are done.

Case 2: $\{a\}=\{c, d\}$ and $\{a, b\}=\{c\}$. Here $c \in \{c, d\}$ and $\{c, d\}=\{a\}$,so $c \in \{a\}$; thus $c = a$; analogously, $d = a$. Also, $b \in \{a, b\}$ and $\{a, b\}=\{c\}$,so $b \in \{c\}$; hence $b = c$. Thus $a = b = c = d$, and we are done. ■

1.31 Definition The *Cartesian product* of two classes *A* and *B* is the class of all ordered pairs *(x, y)* where $x \in A$ and $y \in B$. In symbols,

$$A \times B = \{(x, y) : x \in A \quad \text{and} \quad y \in B\}.$$

The following are a few simple properties of Cartesian products.

1.32 Theorem For all classes *A*, *B*, and *C*,

i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

iii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof

$$\begin{aligned}
\text{i) } (x, y) \in A \times (B \cap C) &\Leftrightarrow x \in A \quad \text{and} \quad y \in B \cap C \\
&\Leftrightarrow x \in A \quad \text{and} \quad y \in B \quad \text{and} \quad y \in C \\
&\Leftrightarrow (x, y) \in A \times B \quad \text{and} \quad (x, y) \in A \times C \\
&\Leftrightarrow (x, y) \in (A \times B) \cap (A \times C).
\end{aligned}$$

$$\begin{aligned}
\text{iii) } (x, y) \in (A \times B) \cap (C \times D) &\Leftrightarrow (x, y) \in A \times B \quad \text{and} \quad (x, y) \in C \times D \\
&\Leftrightarrow x \in A \text{ and } y \in B \text{ and } x \in C \text{ and } y \in D \\
&\Leftrightarrow x \in A \cap C \quad \text{and} \quad y \in B \cap D \\
&\Leftrightarrow (x, y) \in (A \cap C) \times (B \cap D). \blacksquare
\end{aligned}$$

Just as we found it instructive to represent relations between classes by means of Venn diagrams, it is often convenient to illustrate relations between products of classes by using a graphic device known as a *coordinate diagram*. A coordinate diagram is analogous to the familiar Cartesian coordinate plane; there are two axes— a vertical one and a horizontal one—but we consider only one "quadrant." If we wish to represent a class *A* × *B*, then a segment of the horizontal axis is marked off to represent *A* and a segment of the vertical axis is marked off to represent *B*; *A* × *B* is the rectangle determined by these two segments (Fig. 5). As an example of the use of coordinate diagrams, Theorem 1.32(iii) is illustrated in Fig. 6.

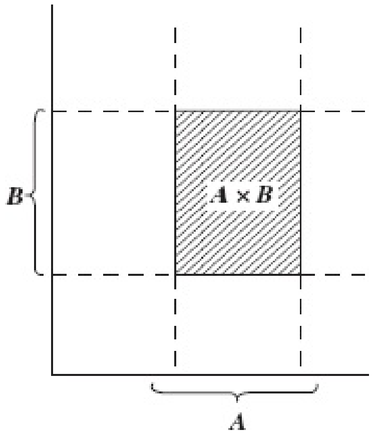


Fig.5

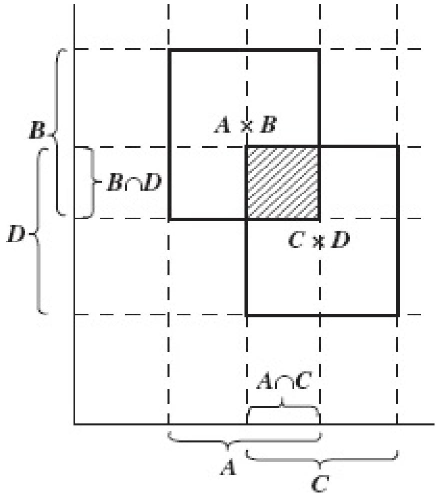


Fig.6

EXERCISES 1.4

1. Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, $C = \{x, y, z\}$. Find $A \times B$, $B \times A$, $C \times (B \times A)$, $(A \cup B) \times C$, $(A \times C) \cup (B \times C)$, $(A \cup B) \times (B \cup C)$.
2. Prove Theorem 1.32(ii).
3. Prove that $A \times (B - D) = (A \times B) - (A \times D)$.
4. Prove that $(A \times B) \cap (C \times D) = (A \times D) \cap (C \times B)$.
5. If A, B and C are classes, prove the following.
   a) $(A \times A) \cap (B \times C) = (A \cap B) \times (A \cap C)$.
   b) $(A \times B) - (C \times C) = [(A - C) \times B] \cup [A \times (B - C)]$.
   c) $(A \times A) - (B \times C) = [(A - B) \times A] \cup [A \times (A - C)]$.
6. Prove that A and B are disjoint if and only if, for any nonempty class C, $A \times C$ and $B \times C$ are disjoint.
7. If A and C are nonempty classes, prove that $A \subseteq B$ ad $C \subseteq D$ if and only if $A \times C \subseteq B \times D$.
8. Let A, B, C, D be nonempty classes. Prove that $A \times B = C \times D$ if and only if $A = C$ and $B = D$.
9. If A, B, and C are any classes, prove
   a) $A \times B$ and $A' \times C$ are disjoint,   b) $B \times A$ and $C \times A'$ are disjoint.
10. Prove that $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.
11. Prove each of the following.
    a) If $a = \{b\}$, then $b \in a$.
    b) $x = y$ if and only if $\{x\} = \{y\}$.
    c) $x \in a$ if and only if $\{x\} \subseteq a$.
    d) $\{a, b\} = \{a\}$ if and only if $a = b$.
12. We give the following alternative definition of ordered pairs:
    $(x, y) = \{\{x, \emptyset\}, \{y, \{\emptyset\}\}\}$. Using this definition, prove that

$$(a, b) = (c, d) \Rightarrow a = c \text{ and } b = d.$$

5 GRAPHS

A class of ordered pairs is called a *graph*. In other words, a graph is an arbitrary subclass of $\mathcal{U} \times \mathcal{U}$.

The importance of graphs will become apparent to the reader in Chapters 2 and 3. It may be shown, for instance, that a function from A to B is a graph $G \subseteq A \times B$ with certain special properties. Specifically, G consists of all the pairs (x, y) such that $y = f(x)$. This example may help to motivate the following definitions.

1.33 Definition If G is a graph, then G^{-1} is the graph defined by

$$G^{-1} = \{(x, y) : (y, x) \in G\}.$$

1.34 Definition If G and H are graphs, then $G \circ H$ is the graph defined as follows:

$$G \circ H = \{(x, y) : \exists z \ni (x, z) \in H \quad \text{and} \quad (z, y) \in G\}.$$

The following are a few basic properties of graphs.

1.35 Theorem If G, H, and J are graphs, then the following statements hold:

i) $(G \circ H) \circ J = G \circ (H \circ J)$.
ii) $(G^{-1})^{-1} = G$.
iii) $(G \circ H)^{-1} = H^{-1} \circ G^{-1}$.

Proof

i)
$$\begin{aligned}(x, y) \in (G \circ H) \circ J &\Leftrightarrow \exists z \ni (x, z) \in J \quad \text{and} \quad (z, y) \in G \circ H \\ &\Leftrightarrow \exists w \text{ and } \exists z \ni (x, z) \in J \quad \text{and} \quad (z, w) \in H \\ &\quad\ \text{and} \quad (w, y) \in G \\ &\Leftrightarrow \exists w \ni (x, w) \in H \circ J \quad \text{and} \quad (w, y) \in G \\ &\Leftrightarrow (x, y) \in G \circ (H \circ J).\end{aligned}$$

ii)
$$\begin{aligned}(x, y) \in (G^{-1})^{-1} &\Leftrightarrow (y, x) \in G^{-1} \\ &\Leftrightarrow (x, y) \in G.\end{aligned}$$

iii)
$$\begin{aligned}(x, y) \in (G \circ H)^{-1} &\Leftrightarrow (y, x) \in G \circ H \\ &\Leftrightarrow \exists z \ni (y, z) \in H \quad \text{and} \quad (z, x) \in G \\ &\Leftrightarrow \exists z \ni (x, z) \in G^{-1} \quad \text{and} \quad (z, y) \in H^{-1} \\ &\Leftrightarrow (x, y) \in H^{-1} \circ G^{-1}. \ \blacksquare\end{aligned}$$

1.36 Definition Let G be a graph. By the *domain* of G we mean the class

$$\text{dom } G = \{x : \exists y \ni (x, y) \in G\},$$

and by the *range* of G we mean the class

$$\text{ran } G = \{y : \exists x \ni (x, y) \in G\}.$$

In other words, the domain of G is the class of all “first components” of elements of G, and the range of

G is the class of all “second components” of element of G.

1.37 Theorem If G and H are graphs, then

$$\begin{array}{llll} \text{i)} & \operatorname{dom} G = \operatorname{ran} G^{-1}, & \text{ii)} & \operatorname{ran} G = \operatorname{dom} G^{-1}, \\ \text{iii)} & \operatorname{dom}(G \circ H) \subseteq \operatorname{dom} H, & \text{iv)} & \operatorname{ran}(G \circ H) \subseteq \operatorname{ran} G. \end{array}$$

Proof

$$\begin{aligned} \text{i)}\quad x \in \operatorname{dom} G &\Leftrightarrow \exists y \ni (x, y) \in G \\ &\Leftrightarrow \exists y \ni (y, x) \in G^{-1} \\ &\Leftrightarrow x \in \operatorname{ran} G^{-1}. \end{aligned}$$

$$\begin{aligned} \text{iii)}\quad x \in \operatorname{dom}(G \circ H) &\Rightarrow \exists y \ni (x, y) \in (G \circ H) \\ &\Rightarrow \exists z \ni (x, z) \in H \quad \text{and} \quad (z, y) \in G \\ &\Rightarrow x \in \operatorname{dom} H. \blacksquare \end{aligned}$$

1.38 Corollary Let G and H be graphs. If ran $H \subseteq$ dom G then dom $G \circ H$= dom H.

The proof of this theorem is left as an exercise for the reader.

EXERCISES 1.5

1. Let

$$G = \{(b, b), (b, c), (c, c), (c, d)\}$$

and

$$H = \{(b, a), (c, b), (d, c)\}.$$

Find $G^{-1}, H^{-1}, G \circ H, H \circ G, (G \circ H)^{-1}, (G \cup H)^{-1}, H^{-1} \circ G$.

2. Prove Theorem 1.37, parts (ii) and (iv).
3. Prove Theorem 1.38.
4. If G, H, and J are graphs, prove each of the following.

   a) $(H \cup J) \circ G = (H \circ G) \cup (J \circ G)$, b) $(G - H)^{-1} = G^{-1} - H^{-1}$,
   c) $G \circ (H \cap J) \subseteq (G \circ H) \cap (G \circ J)$, d) $(G \circ H) - (G \circ J) \subseteq G \circ (H - J)$.
5. If G and H are graphs, prove each of the following.

   a) $(G \cap H)^{-1} = G^{-1} \cap H^{-1}$, b) $(G \cup H)^{-1} = G^{-1} \cup H^{-1}$.
6. If G, H, J, and K are graphs, prove

   a) if $G \subseteq H$ and $J \subseteq K$, then $G \circ J \subseteq H \circ K$,

   b) $G \subseteq H$ if and only if $G^{-1} \subseteq H^{-1}$.
7. If A, B, and C are classes, prove each of the following.

a) $(A \times B)^{-1} = B \times A$.

b) If $A \cap B \neq \emptyset$, then $(A \times B) \circ (A \times B) = A \times B$.

c) If A and B are disjoint, then $(A \times B) \circ (A \times B) = \emptyset$.

d) If $B \neq \emptyset$, then $(B \times C) \circ (A \times B) = A \times C$.

8. Let G and H be graphs; prove each of the following.

a) If $G \subseteq A \times B$, then $G^{-1} \subseteq B \times A$.

b) If $G \subseteq A \times B$ and $H \subseteq B \times C$, then $H \circ G \subseteq A \times C$.

9. If G and H are graphs, prove each of the following.

a) dom$(G \cup H) = ($dom $G) \cup ($dom $H)$.

b) ran$(G \cup H) = ($ran $G) \cup ($ran $H)$.

c) dom $G -$ dom $H \subseteq$ dom$(G - H)$.

d) ran $G -$ ran $H \subseteq$ ran$(G - H)$.

10. Let G be a graph, and let B be a subclass of the domain of G. By the *restriction of G to B* we mean the graph

$$G_{[B]} = \{(x, y) : (x, y) \in G \text{ and } x \in B\}.$$

Prove each of the following.

a) $G_{[B]} = G \cap (B \times \operatorname{ran} G)$, b) $G_{[B \cup C]} = G_{[B]} \cup G_{[C]}$,

c) $G_{[B \cap C]} = G_{[B]} \cap G_{[C]}$, d) $(G \circ H)_{[B]} = G \circ H_{[B]}$.

11. Let G be a graph and let B be a subclass of the domain of G. We use the symbol $G(B)$ to designate the class

$$G(B) = \{y : \exists x \in B \ni (x, y) \in G\}.$$

Prove each of the following.

a) $G(B) = \operatorname{ran} G_{[B]}$, b) $G(B \cup C) = G(B) \cup G(C)$,

c) $G(B \cap C) = G(B) \cap G(C)$, d) If $B \subseteq C$, then $G(B) \subseteq G(C)$.

6 GENERALIZED UNION AND INTERSECTION

Consider the class $\{A_1, A_2, \ldots, A_n\}$; its elements are indexed by the numbers 1, 2, …, n. Such a class if often called an indexed family of classes; the numbers 1, 2, …, n are called indices and the class $\{1, 2, \ldots, n\}$ is called the index class.

More generally, we are frequently led to think of a class I whose elements $i,j,k, \ldots$ serve as indices to designate the elements of a class $\{A_i,A_j, A_k, \ldots\}$. The class $\{A_i,A_j, A_k, \ldots\}$ is called an *indexed family of classes*, I is called its *index class*, and the elements of I are called *indices*. A compact notation which is often used to designate the class $\{A_i, A_j, A_k, \ldots\}$ is

$$\{A_i\}_{i \in I}.$$

Thus, speaking informally, $\{A_i\}_{i \in I}$ is the class of all the classes A_i,as i ranges over I.

Remark. The definition of an indexed family of classes which we have just given is, admittedly, an intuitive one; it relies on the intuitive notion of *indexing*. This intuitive definition is adequate at the present time; however, for future reference, we now give a *formal* definition of the same concept:

> By an indexed family of classes, $\{A_i\}_{i\in I}$, we mean a graph G whose domain is I; for each $i \in I$ we define A_i by

$$A_i = \{x : (i, x) \in G\}.$$

For example, consider $\{A_i\}_{i\in I}$ where $I = \{1, 2\}$, $A_1 = \{a, b\}$, and $A_2 = \{c, d\}$. Then, *formally,* $\{A_i\}_{i\in I}$ is the graph

$$G = \{(1, a), (1, b), (2, c), (2, d)\}.$$

If $\{A_i\}_{i\in I}$ is an indexed family of classes such that for each $i \in I$, A_i is an element, then we let $\{A_i : i \in I\}$ designate the class whose elements are all the A_i, that is, $\{A_i : i \in I\} = \{x : x = A_i$, for some $i \in I\}$. However, we shall follow current mathematical usage and use the two expressions, $\{A_i\}_{i\in I}$ and $\{A_i : i \in I\}$, interchangeably.

1.39 Definition Let $\{A_i\}_{i\in I}$ be an indexed family of classes. The *union of the classes* A_i consists of all the elements which belong to at least one class A_i of the family. In symbols,

$$\bigcup_{i\in I} A_i = \{x : \exists j \in I \ni x \in A_j\}.$$

The *intersection of the classes* A_i consists of all the elements which belong to every class A_i of the family. In symbols,

$$\bigcap_{i\in I} A_i = \{x : \forall i \in I, x \in A_i\}.$$

The following are some basic properties of indexed families of classes.

1.40 Theorem Let $\{A_i\}_{i\in I}$ be an indexed family of classes.

i) If $A_i \subseteq B$ for every $i \in I$, then $\bigcup_{i\in I} A_i \subseteq B$.

ii) If $B \subseteq A_i$ for every $i \in I$, then $B \subseteq \bigcap_{i\in I} A_i$.

Proof

i) Suppose that $A_i \subseteq B$ for every $i \in I$ $x \in \bigcup_{i\in I} A_i$;, then $x \in A_j$ for some $j \in I$; but $A_j \subseteq B$,so $x \in B$. Thus $\bigcup_{i\in I} A_i \subseteq B$.

The proof of (ii) is left as an exercise for the reader. ■

1.41 Theorem (*Generalized deMorgan's Laws*). Let $\{A_i\}_{i \in I}$ be an index family of classes. Then,

i) $(\bigcup_{i \in I} A_i)' = \bigcap_{i \in I} A_i'$.

ii) $(\bigcap_{i \in I} A_i)' = \bigcup_{i \in I} A_i'$.

Proof

i)
$$\begin{aligned} x \in (\bigcup_{i \in I} A_i)' &\Leftrightarrow x \notin \bigcup_{i \in I} A_i \\ &\Leftrightarrow \forall i \in I, x \notin A_i \\ &\Leftrightarrow \forall i \in I, x \in A_i' \\ &\Leftrightarrow x \in \bigcap_{i \in I} A_i'. \end{aligned}$$

The proof of (ii) is left as an exercise for the reader. ■

1.42 Theorem (*Generalized Distributive Laws*). Let $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ be indexed families of classes. Then

i) $(\bigcup_{i \in I} A_i) \cap (\bigcup_{j \in J} B_j) = \bigcup_{(i,j) \in I \times J} (A_i \cap B_j)$,

ii) $(\bigcap_{i \in I} A_i) \cup (\bigcap_{j \in J} B_j) = \bigcap_{(i,j) \in I \times J} (A_i \cup B_j)$.

Proof

i)
$$\begin{aligned} x \in (\bigcup_{i \in I} A_i) \cap (\bigcup_{j \in J} B_j) &\Leftrightarrow x \in \bigcup_{i \in I} A_i \quad \text{and} \quad x \in \bigcup_{j \in J} B_j \\ &\Leftrightarrow x \in A_h \text{ for some } h \in I \text{ and } x \in B_k \text{ for some } k \in J \\ &\Leftrightarrow x \in A_h \cap B_k \text{ for some } (h, k) \in I \times J \\ &\Leftrightarrow x \in \bigcup_{(i,j) \in I \times J} (A_i \cap B_j). \end{aligned}$$

The proof of (ii) is left as an exercise for the reader. ■

A theorem concerning the union of graphs will be useful to us in the next chapter.

1.43 Theorem Let $\{G_i\}_{i \in I}$ be a family of graphs. Then

i) $\operatorname{dom}(\bigcup_{i \in I} G_i) = \bigcup_{i \in I} (\operatorname{dom} G_i)$.

ii) $\operatorname{ran}(\bigcup_{i \in I} G_i) = \bigcup_{i \in I} (\operatorname{ran} G_i)$.

Proof

i)

$$\begin{aligned} x \in \operatorname{dom}\Big(\bigcup_{i\in I} G_i\Big) &\Leftrightarrow \exists y \ni (x, y) \in \bigcup_{i\in I} G_i && \text{by 1.36} \\ &\Leftrightarrow \exists y \ni (x, y) \in G_j \text{ for some } j \in I && \text{by 1.39} \\ &\Leftrightarrow x \in \operatorname{dom} G_j \text{ for some } j \in I && \text{by 1.36} \\ &\Leftrightarrow x \in \bigcup_{i\in I} (\operatorname{dom} G_i) && \text{by 1.39.} \end{aligned}$$

The proof of (ii) is left as an exercise for the reader. ■

A variant notation for the union and intersection of a family of classes is sometimes useful. If $\mathscr{A}$ is a class (its elements are necessarily classes), we define the *union of* $\mathscr{A}$, or *union of the elements of* $\mathscr{A}$, to be the union of all the classes which are elements for $\mathscr{A}$. In symbols,

1.44 $$\bigcup_{A\in\mathscr{A}} A = \{x : x \in A \text{ for some } A \in \mathscr{A}\}.$$

In other words, $x \in \bigcup_{A\in\mathscr{A}} A$ *A* if and only if there is a class *A* such that *x* ∈ *A* and *A* ∈ $\mathscr{A}$. Analogously, we define the *intersection of* $\mathscr{A}$, *or intersection of the elements of* $\mathscr{A}$, to be the intersection of all the classes which are elements of $\mathscr{A}$. In symbols,

1.45 $$\bigcap_{A\in\mathscr{A}} A = \{x : x \in A \text{ for every } A \in \mathscr{A}\}.$$

1.46 Example Let $\mathscr{A}$={*K*, *L*, *M*}, where *K* ={*a*, *b*, *d*}, *L* ={*a*, *c*, *d*}, and *M* = {*d*, *e*}. Then

$$\bigcup_{A\in\mathscr{A}} A = \{a, b, c, d, e\} \quad \text{and} \quad \bigcap_{A\in\mathscr{A}} A = \{d\}.$$

1.47 *Remark.* It is frequent practice, in the literature of set theory, to write

$$\bigcup \mathscr{A} \quad \text{for} \quad \bigcup_{A\in\mathscr{A}} A$$

and

$$\bigcap \mathscr{A} \quad \text{for} \quad \bigcap_{A\in\mathscr{A}} A.$$

We shall occasionally follow that practice in this book.

EXERCISES 1.6

1. Prove Theorem 1.40(ii).
2. Prove Theorem 1.41(ii).
3. Prove Theorem 1.42(ii).
4. Prove Theorem 1.43(ii).

5. Let $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ be two families of classes with the same index class I. Suppose that $\forall i \in I$, $A_i \subseteq B_i$; prove that
   a) $\bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I} B_i$, b) $\bigcap_{i\in I} A_i \subseteq \bigcap_{i\in I} B_i$.
6. Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be indexed families of classes. Prove the following.
   a) $(\bigcap_{i\in I} A_i) \times (\bigcap_{j\in J} B_j) = \bigcap_{(i,j)\in I\times J} (A_i \times B_j)$,
   b) $(\bigcup_{i\in I} A_i) \times (\bigcup_{j\in J} B_j) = \bigcup_{(i,j)\in I\times J} (A_i \times B_j)$.
7. Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be indexed families of classes. Suppose that $\forall i \in I$, $\exists j \in J$ $B_j \subseteq A_i$. Prove that

$$\bigcap_{j\in J} B_j \subseteq \bigcap_{i\in I} A_i.$$

8. Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be indexed families of classes. Prove that
   a) $(\bigcup_{i\in I} A_i) - (\bigcup_{j\in J} B_j) = \bigcup_{i\in I}(\bigcap_{j\in J}[A_i - B_j])$,
   b) $(\bigcap_{i\in I} A_i) - (\bigcap_{j\in J} B_j) = \bigcap_{i\in I}(\bigcup_{j\in J}[A_i - B_j])$.
9. We say that an indexed family $\{B_i\}_{i\in I}$ is a *covering* of A if $A \subseteq \bigcup_{i\in I} B_i$ Suppose that $\{B_i\}_{i\in I}$ and $\{C_j\}_{j\in J}$ are two distinct coverings of A. Prove that the family $\{(B_i \cap C_j)\}_{(i,j)\in I\times J}$ is a covering of A.
10. Let $a = \{u, v, w\}$, $b = \{w, x\}$, $c = \{w, y\}$, $r = \{a, b\}$, $s = \{b, c\}$, and $p = \{r, s\}$. Find the classes $\cup(\cup p)$, $\cap(\cap p)$, $\cup(\cap p)$, $\cap(\cup p)$.
11. Prove that $\cap(\mathscr{A} \cup \mathscr{B}) = (\cap\mathscr{A}) \cap (\cap\mathscr{B})$.
12. Prove each of the following.
    a) If $A \in \mathscr{B}$, then $A \subseteq \cup\mathscr{B}$ and $\cap\mathscr{B} \subseteq A$.
    b) $\mathscr{A} \subseteq \mathscr{B}$, if and only if $\cup\mathscr{A} \subseteq \cup\mathscr{B}$.
    c) If $\varnothing \in \mathscr{A}$, then $\cap\mathscr{A} = \varnothing$.

7 SETS

Undoubtedly, everything presented in this chapter is fairly familiar to you. Even though we used the word *class* where you are more accustomed to hearing *set*,it is obvious that the “union” and “intersection” defined in this chapter are the same as the familiar union and intersection of sets. The Cartesian product of classes is no different from the Cartesian product of sets, and the same is true for the other concepts introduced in this chapter. This has been very convenient, but it is time to face the fact that if we fail to distinguish between the two kinds of classes—namely sets and proper classes—we shall rapidly find ourselves in the midst of logical contradictions.

There are two kinds of classes: Sets and proper classes.

1.48 Definition If $X \in Y$ for some class Y, then X is a *set*.

If $X \notin Y$ for any class Y, then X is a proper class

You will notice immediately that a set is exactly an element. This is the awkward

aspect of our terminology: It may irk us, but axiomatic set theory requires us to accept the words “element” and “set” as synonyms, to be distinguished from proper classes, which are not sets, and cannot be elements of anything.

Proper classes are a little like embarrassing relatives that we are forced to acknowledge, but we'd rather keep at arm's length. Those are the classes that may lead to paradox, and in order to do mathematics safely we want to be sure that we are dealing only with sets. So the most important axioms of set theory are designed to provide the assurance that when we carry out operations on sets—for example when we form a generalized union of sets—we are not inadvertently creating a proper class. To be precise, we want a guarantee that when performing operations on sets, the results of the operations are sets and not proper classes.

Since, intuitively, a set is any class that is not "too large", we would certainly expect every subclass of a set to be a set. So if A is not "too large" and $B \subseteq A$, then B should not be "too large". We state this as an axiom.

A3. Every subclass of a set is a set.

Axiom A3 has a simple consequence: From Theorem 1.20, $A \cap B \subseteq A$. So by Axiom A3, if A is a set, then $A \cap B$ is a set. That is, *the intersection of any two sets is a set*.

With all this talk about sets, one would assume that we could exhibit one— in other words, we could say, "Here, this is a set!" The fact is that we can't: Our definitions and axioms so far have not given us any sets. So we must now state an axiom whose only purpose is to assert that sets exist—well, at least one set exists.

A4. $\varnothing$ is a set.

Mighty oaks from little acorns grow. We shall see later that from the humble empty set, many more sets can be shown to exist. As a preview, consider the set $\{\varnothing\}$: It has one element, namely the empty set. The set $\{\varnothing, \{\varnothing\}\}$ has two elements. We'll stop here for now. Actually, in a later chapter we present another axiom that provides for the existence of sets, this time of infinite sets. After adding that axiom, our Axiom A4 will be redundant. Meanwhile it will serve us well.

It is reasonable to assume that if a and b are sets, then the doubleton $\{a, b\}$, with only two elements, is not too large to be a set.

A5. If a and b are sets, then $\{a, b\}$ is a set.

From Axioms A3 and A5, it follows immediately that if a is a set, then the singleton $\{a\}$ is a set. The next two axioms are especially important, because they guarantee that if you combine sets into larger sets—for example by forming the generalized union of a family of sets, or a Cartesian product of many sets—these larger collections are still sets.

1.49 Definition Let A be a set; by the *power set* of A we mean the class of all the subsets of A. In symbols, the power set of A is the class

$$\mathscr{P}(A) = \{B : B \subseteq A\}.$$

Note that by Axiom A3, $\mathscr{P}(A)$is the class of all the *sets* B which satisfy $B \subseteq A$.

1.50 Example If $A = \{a, b\}$, then

$$\mathscr{P}(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}.$$

Note that $B \in \mathscr{P}(A)$if and only if $B \subseteq A$.

It is easy to see that $\mathscr{P}(A)$ is a larger class than A, for $\mathscr{P}(A)$ includes (among other things) all the singletons $\{x\}$ as x ranges over A. Thus we may legitimately ask the following question: if A is a set, is it necessarily true that $\mathscr{P}(A)$ is a set? Or is it possible that $\mathscr{P}(A)$may be "too large" to be a set? An analogous question may be raised in regard to the union of sets: if $\mathscr{A}$ is a set of sets, $\bigcup_{A \in \mathscr{A}} A$is a set, or might it be "too large" to be a set? These questions may be answered *intuitively* as follows: none of the "giant" collections which cause contradictions in intuitive set theory can be obtained either as a power set of a set or as a union of a set of sets. Thus we are justified in adopting the following as axioms.

A6. If $\mathscr{A}$ is a set of sets, then $\bigcup_{A \in \mathscr{A}} A$ is a set.

A7. If A is a set, then power set of A is a set.

If A and B are sets, then by Axiom A5, $\{A, B\}$ is a set; it follows immediately from Definition 1.44 that $\bigcup_{X \in \{A, B\}} X = A \cup B$; thus, by Axiom A6, $A \cup B$ is a set. This shows that *the union of two sets is a set*.

Several other axioms for sets have been proposed, but are not essential in everyday mathematical practice. One axiom, that we shall encounter again in the final chapter of this book, is called the *Axiom of Foundation*. It states the following:

A8. If A is any set, there is an element $a \in A$ such that $a \cap A = \varnothing$.
This axiom has an equivalent form which has applications in Chapter 11:
Any descending sequence of sets $\ldots \in A_4 \in A_3 \in A_2 \in A_1$ is finite. In other words, you cannot have an infinite descending sequence of sets, each an element of the previous one.

A very intriguing axiom, which has surprisingly far-reaching consequences, is the following, which we do not include in the axiomatic system of this book: Every proper class is in one-to-one correspondence with the universal class $\mathscr{U}$, that is, with the class of all sets. Though we shall not adopt this axiom here, it helps us to form an intuitive image of what proper classes are like: They are classes whose size is as large as that of the class containing *all* sets.

Three more axioms, whose purpose will be explained later, are introduced in Chapters 5, 6 and 7.

1.51 Theorem If A and B are sets, then $A \times B$ is a set.

Proof. Let A and B be sets. By Axiom A6, $A \cup B$ is a set; by Axiom A7, $\mathscr{P}(A \cup B)$ is a set; finally, by Axiom A7 again, $\mathscr{P}[\mathscr{P}(A \cup B)]$ is a set. We will prove that $A \times B \subseteq \mathscr{P}[\mathscr{P}(A \cup B)]$, and it will follow, by Axiom A3, that $A \times B$ is a set.

Let $(x, y) \in A \times B$. By 1.29, $(x, y) = \{\{x\}, \{x, y\}\}$. Now $x \in A \cup B$, hence $\{x\} \subseteq A \cup B$,so $\{x\} \in \mathscr{P}(A \cup B)$. Similarly, $x \in A \cup B$ and $y \in A \cup B$,so $\{x, y\} \subseteq A \cup B$, hence $\{x, y\} \in \mathscr{P}(A \cup B)$. We have just shown that $\{x\}$ and $\{x, y\}$ are elements of $\mathscr{P}(A \cup B)$, hence

$$\{\{x\}, \{x, y\}\} \subseteq \mathscr{P}(A \cup B);$$

it follows that

$$\{\{x\}, \{x, y\}\} \in \mathscr{P}[\mathscr{P}(A \cup B)],$$

thus is,

$$(x, y) \in \mathscr{P}[\mathscr{P}(A \cup B)].$$

Thus

$$A \times B \subseteq \mathscr{P}[\mathscr{P}(A \cup B)]. \blacksquare$$

It follows from Theorem 1.51 and Axiom A3 that if A and B are sets, then any graph $G \subseteq A \times B$ is a set.

It is easy to show that if G is a set, then dom G and ran G are sets (see Exercise 5, Exercise Set 1.7). Using this fact, one can easily show that if G and H are sets, then $G \circ H$ and G^{-1} are sets (see Exercise 6, Exercise Set 1.7).

EXERCISES 1.7

1. If A and B are sets, prove that $A - B$ and $A + B$ are sets. (See Exercise 8, Exercise Set 1.3.)
2. If A is a proper class and $A \subseteq B$, prove that B is a proper class. Conclude that the union of two proper classes is a proper class.
3. Prove that the "Russell class" and the universal class are proper classes. [*Hint*. Use the result of Exercise 8, Exercise Set 1.2.]
4. Let $\{A_i\}_{i \in I}$ be an indexed family of sets. Prove that $\bigcap_{i \in I} A_i$ is a set.
5. Let G be a graph. Prove that if G is a set, then dom G and ran G are sets. [*Hint*: Show that both dom G and ran G are subsets of $\cup(\cup G)$.]
6. Let G and H be graphs. Prove that if G and H are sets, then G^{-1} and $G \circ H$ are sets.
7. Let $r = \{a, b\}$, $s = \{b, c\}$, $p = \{r, s\}$. Find the sets $\mathscr{P}(r)$, $\mathscr{P}(\mathscr{P}(r))$, and $\mathscr{P}(\cup p)$.
8. Let A and B be sets; prove the following.
   a) $A \subseteq B$ if and only if $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.
   b) $A = B$ if and only if $\mathscr{P}(A) = \mathscr{P}(B)$.
   c) $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$.
   d) $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$.
   e) $A \cap B = \emptyset$ if only if $\mathscr{P}(A) \cap \mathscr{P}(B) = \{\emptyset\}$.
9. If A and $\mathscr{B}$ are sets, prove the following.
   a) $\cup(\mathscr{P}(\mathscr{B})) = \mathscr{B}$, b) $\cap(\mathscr{P}(\mathscr{B})) = \emptyset$,
   c) If $\mathscr{P}(A) \in \mathscr{P}(\mathscr{B})$ then $A \in \mathscr{B}$.
10. Exhibit the sets $\mathscr{P}(\mathscr{P}(\emptyset))$ and $\mathscr{P}[\mathscr{P}(\mathscr{P}(\emptyset))]$.

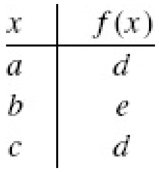
2

Functions

1 INTRODUCTION

The concept of a function is one of the most basic mathematical ideas and enters into almost every mathematical discussion. A function is generally defined as follows: If A and B are classes, then a function from A to B is a rule which to every element $x \in A$ assigns a unique element $y \in B$; to indicate this connection between x and y we usually write $y = f(x)$. For instance, consider the function $y = \sin x$; if we take A to be the set of all the real numbers and B to be the closed interval $[-1, 1]$, then it is easy to see that $y = \sin x$ is a rule which, to every number $x \in A$, assigns a unique number $y \in B$.

The graph of a function is defined as follows: If f is a function from A to B, then the graph of f is the class of all ordered pairs (x, y) such that $y = f(x)$. For example, let $A = \{a, b, c\}$ and $B = \{d, e\}$, and let f be the function defined by the following table.



x	$f(x)$
a	d
b	e
c	d

The graph of f is $\{(a, d), (b, e), (c, d)\}$.

Clearly, we may use the information contained in the table to construct the graph of f ; we may also operate the other way, that is, we may use the information contained in the graph to construct the table of f . Thus a function f completely determines its graph, and conversely, its graph completely determines f . Hence there is no need to distinguish between a function and its graph.

Since a function and its graph are essentially one and the same thing, we may, if we wish, *define* a function to be a graph. There is an important advantage to be gained by doing this—namely, we avoid having to introduce the word *rule* as a new undefined concept of set theory. For this reason it is customary, in rigorous treatments of mathematics, to introduce the notion of *function* via that of *graph*. We shall follow that procedure here.

2 FUNDAMENTAL CONCEPTS AND DEFINITIONS

We begin by giving our “official” definition of a function.

2.1 Definition A *function from A to B* is a triple of objects $\langle f, A, B \rangle$, where A and B are classes and f is a subclass of $A \times B$ with the following properties.

F1. $\forall x \in A, \exists y \in B$ such that $(x, y) \in f$.
F2. If $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.

It is customary to write $f : A \to B$ instead of $\langle f, A, B \rangle$.

In ordinary mathematical applications, every function $f : A \to B$ is a function from a *set* A to a *set* B. However, the intuitive concept of a function from A to B is meaningful for any two collections A and B, whether A and B be sets or proper classes; hence it is natural to give the definition of a function in its most general form, letting A and B be any classes. Once again, every set is a class, hence everything we have to say about functions from a class A to a class B applies, in particular, to functions from a set A to a set B.

Let $f : A \to B$ be a function; if $(x, y) \in f$, we say that *y is the image of x* (with respect to f); we also say that *x is the pre-image of y* (with respect to f); we also say that *f maps x onto y*, and symbolize this statement by $x \xmapsto{f} y$. (The reader may, if he wishes, picture these statements as in Fig. 1.)

Thus, F1 states that

every element $x \in A$ has an image $y \in B$.

F2 states that if $x \in A$, then

the image of x is unique;

for if $(x, y_1) \in f$ and $(x, y_2) \in f$, that is, if y_1 and y_2 are both images of x, then F2 dictates that $y_1 = y_2$. It follows that F1 and F2 combined state that

2.2 Every element $x \in A$ has a uniquely determined image $y \in B$.

2.3 Theorem Let A and B be classes and let f be a graph. Then $f : A \to B$ is a function if and only if

i) F2 holds,

ii) $\operatorname{dom} f = A$, and

iii) $\operatorname{ran} f \subseteq B$.

Proof. Suppose $f : A \to B$ is a function; by 2.1, F2 holds. Furthermore,

$$\begin{aligned}
&\text{a) } x \in \operatorname{dom} f \Rightarrow \exists y \ni (x, y) \in f && \text{by 1.36} \\
&\qquad\qquad\qquad \Rightarrow (x, y) \in A \times B && \text{because } f \subseteq A \times B \text{ by 2.1} \\
&\qquad\qquad\qquad \Rightarrow x \in A && \text{by 1.31.} \\
&\text{b) } x \in A \Rightarrow \exists y \in B \ni (x, y) \in f && \text{by F1} \\
&\qquad\qquad \Rightarrow x \in \operatorname{dom} f && \text{by 1.36.} \\
&\text{c) } y \in \operatorname{ran} f \Rightarrow \exists x \ni (x, y) \in f && \text{by 1.36} \\
&\qquad\qquad\qquad \Rightarrow (x, y) \in A \times B && \text{because } f \subseteq A \times B \text{ by 2.1} \\
&\qquad\qquad\qquad \Rightarrow y \in B && \text{by 1.31.}
\end{aligned}$$

By (a) and (b), $\operatorname{dom} f = A$; by (c), $\operatorname{ran} f \subseteq B$. Thus, (i), (ii), and (iii) hold.

For the converse, suppose that (i), (ii), and (iii) hold.

$$\begin{aligned}
&\text{a) } (x, y) \in f \Rightarrow x \in \operatorname{dom} f \text{ and } y \in \operatorname{ran} f && \text{by 1.36} \\
&\qquad\qquad\qquad \Rightarrow x \in A \text{ and } y \in B && \text{by (ii) and (iii)} \\
&\qquad\qquad\qquad \Rightarrow (x, y) \in A \times B && \text{by 1.31.}
\end{aligned}$$

Thus, $f \subseteq A \times B$.

b) Let x be an arbitrary element of A. By (ii), $x \in \operatorname{dom} f$; hence $\exists y\ (x, y) \in f$; by $y \in \operatorname{ran} f$, so by (iii), $y \in B$. This proves that F1 holds. By (i), F2 holds; thus, by 2.1, $f : A \to B$ is a function. ■

From Theorem 2.3 we conclude, in particular, that if $f : A \to B$ is a function, then A is the domain of f and B contains the range of f. We call B the *codomain* of $f : A \to B$.

2.4 Corollary Let $f : A \to B$ be a function; if C is any class such that $\operatorname{ran} f \subseteq C$, then $f : A \to C$ is a function.

Proof. If $f : A \to B$ is a function, then by 2.3, F2 holds and $\operatorname{dom} f = A$; thus, if $\operatorname{ran} f \subseteq C$, then, by 2.3, $f : A \to C$ is a function. ■

Let $f : A \to B$ be a function and let $x \in A$; it is customary to use the symbol $f(x)$ to designate the image of x. Thus,

$y = f(x)$ has the same meeting as $(x, y) \in f$.

When we write $y = f(x)$ instead of $(x, y) \in f$, Conditions F1 and F2 take the form

F1. $\forall x \in A, \exists y \in B, y = f(x)$.
F2. If $y_1 = f(x)$ and $y_2 = f(x)$, then $y_1 = y_2$.

It is often convenient to write F2 in a slightly different way. F2 states that if $(x, y_1) \in f$ and $(x, y_2) \in f$, that is, if

$$\begin{tikzcd} & y_1 \\ x \arrow[ur, mapsto, "f"] \arrow[dr, mapsto, "f"'] & \\ & y_2 \end{tikzcd}$$

then $y_1 = y_2$. This is the same as saying that if $x_1 \xmapsto{f} f(x_1)$, $x_2 \xmapsto{f} f(x_2)$ and $x_1 = x_2$, that is, if

$$\begin{tikzcd} & & & f(x_1) \\ x_1 \arrow[urrr, mapsto, "f"] & = & x_2 \arrow[dr, mapsto, "f"] & \\ & & & f(x_2) \end{tikzcd}$$

then $f(x_1) = f(x_2)$. Thus, F2 may be written in the form

F2°. If $x_1 = x_2$, then $f(x_1) = f(x_2)$.

2.5 Theorem Let $f : A \to B$ and $g : A \to B$ be functions. Then $f = g$ if and only if $f(x) = g(x)$, $\forall x \in A$.

Proof. First, let us assume that $f = g$. Then, for arbitrary $x \in A$,

$$y = f(x) \Leftrightarrow (x, y) \in f \Leftrightarrow (x, y) \in g \Leftrightarrow y = g(x);$$

thus, $f(x) = g(x)$.

Conversely, assume that $f(x) = g(x)$, $\forall x \in A$. Then

$$(x, y) \in f \Leftrightarrow y = f(x) \Leftrightarrow y = g(x) \Leftrightarrow (x, y) \in g;$$

thus, $f = g$. ■

Injective, Surjective and Bijective Functions

The following definitions are of great importance in the study of functions.

2.6 Definition A function $f : A \to B$ is said to be *injective* if it has the following property.

INJ. If $(x_1,y) \in f$ and $(x_2,y) \in f$, then $x_1 = x_2$.

The reader should note that INJ states, simply, that if y is any element of B, then

y has no more than one pre-image;

for if $(x_1,y) \in f$ and $(x_2,y) \in f$, that is, if x_1 and x_2 are both pre-images of y, then INJ dictates that $x_1 = x_2$.

It is often convenient to write INJ in a slightly different way. INJ states that if $(x_1,y) \in f$ and $(x_2,y) \in f$, that is, if

$$\begin{tikzcd} x_1 \arrow[dr, "f"] & \\ & y \\ x_2 \arrow[ur, "f"] & \end{tikzcd}$$

then $x_1 = x_2$. This is the same as saying that if x_1 and x_2 are elements of A and $f(x_1) = f(x_2)$, that is, if

$$\begin{tikzcd} x_1 \arrow[dr] & & \\ & f(x_1) \arrow[r, phantom, "="] & f(x_2), \\ x_2 \arrow[urr] & & \end{tikzcd}$$

then $x_1 = x_2$. Thus, INJ may be written in the form

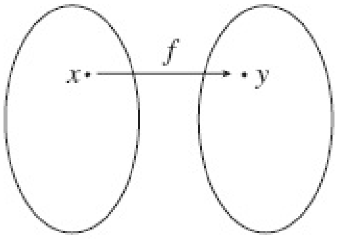
INJ°. If $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(The function of Fig. 2 is injective, whereas the function of Fig. 3 is not injective.)

2.7 Definition A function $f : A \to B$ is said to be *surjective* if it has the following property:

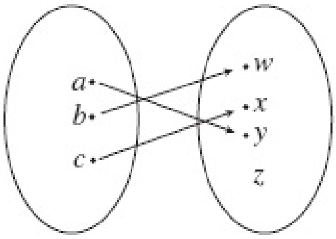
SURJ. $\forall y \in B, \exists x \in A \; y = f(x)$.

Clearly, condition **SURJ** states that *every* element of B is the image of some element of A; that is, $B \subseteq \text{ran } f$. But $\text{ran } f \subseteq B$ by Theorem 2.3; hence **$f : A \to B$ is surjective if and only if $\text{ran } f = B$**. (The function of Fig. 3 is surjective, whereas the function of Fig. 2 is not surjective.)



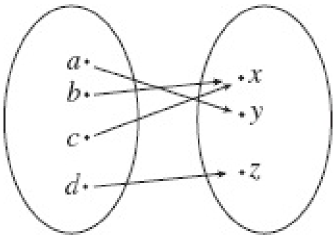
f maps *x* onto *y*
y is the Image of *x*

Fig.1



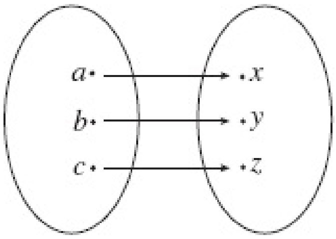
f is injective

Fig.2



f is surjective

Fig.3



f is bijective

Fig.4

2.8 Definition A function $f : A \to B$ is said to be *bijective* if it is both injective and surjective.

To say that $f : A \to B$ is injective is to say that every element of B is the image of *no more than one* element of A; to say that f is surjective is to say that every element of B is the image of *at least one* element of A; thus, to say that f is bijective is to say that every element of B is the image of *exactly one* element of A (Fig. 4). In other words, if $f : A \to B$ is a bijective function, every element of A has exactly one image in B and every element of B has exactly one pre-image in A; thus all the elements of A and all the elements of B are associated in pairs; for this reason, if f is bijective, it is sometimes called a *one-to-one correspondence* between A and B.

2.9 Definition If there exists a bijective function $f : A \to B$, then we say that *A and B are in one-to-one correspondence*.

Examples of Functions

2.10 *Identity function.* Let A be a class; by the *identity function on A* we mean the function $I_A : A \to A$ given by

$$I_A(x) = x, \qquad \forall x \in A.$$

In other words,

$$I_A = \{(x, x) : x \in A\}.$$

I_A is clearly injective, for suppose $I_A(x) = I_A(y)$;now $I_A(x) = x$ and $I_A(y) = y$, so $x = y$; thus INJ° holds. I_A is surjective because, obviously, the range of I_A is A. Thus I_A is bijective.

2.11 *Constant function.* Let A and B be classes, and let b be an element of B. By the *constant function* K_b we mean the function $K_b : A \to B$ given by

$$K_b(x) = b, \qquad \forall x \in A.$$

In other words, $K_b = \{(x, b) : x \in A\}$.

Note that if A has more than one element, K_b is not injective; if B has more than one element, K_b is not surjective.

2.12 *Inclusion function.* Let A be a class and let B be a subclass of A. By the *inclusion function of B in A* we mean the function $E_B : B \to A$ given by

$$E_B(x) = x, \qquad \forall x \in B.$$

Note that if $B = A$, the inclusion function coincides with the identity function I_A. By the argument used in 2.10, E_B is injective; however, if $B \neq A$, then E_B is not surjective.

2.13 *Characteristic function.* Let 2 designate a class of two elements, say the class $\{0, 1\}$. If A is a class and B is a subclass of A, the *characteristic function of B in A* is the function $C_B : A \to 2$ given by

$$C_B(x) = \begin{matrix} 0 & \text{if} & x \in B, \\ 1 & \text{if} & x \notin B, \end{matrix} \qquad \forall x \in A.$$

The C_B maps every element of B onto 0 and every element of $A - B$ onto 1.

2.14 *Restriction of a function.* Let $f : A \to B$ be a function and let C be a subclass of A. By the *restriction* *of f to C* we mean the function $f_{[C]} : C \to B$ given by

$$f_{[C]}(x) = f(x), \qquad \forall x \in C.$$

To put it another way, $f_{[C]} = \{(x, y) : (x, y) \in f \text{ and } x \in C\}$. Note that $f_{[C]} \subseteq f$.

Restrictions of functions have the following properties, which will be useful to us later.

2.15 Theorem If $f : B \cup C \to A$ is a function, then $f = f_{[B]} \cup f_{[C]}$.

The simple proof of this theorem is left as an exercise for the reader.

2.16 Theorem Let $f_1 : B \to A$ and $f_2 : C \to A$ be functions, where $B \cap C = \varnothing$. If $f = f_1 \cup f_2$, then the following hold:

i) $f : B \cup C \to A$ is a function.

ii) $f_1 = f_{[B]}$ and $f_2 = f_{[C]}$.

iii) If $x \in B$ then $f(x) = f_1(x)$, and if $x \in C$ then $f(x) = f_2(x)$.

Proof. We will begin by proving the following two relations.

a) $(x, y) \in f$ and $x \in B \Leftrightarrow (x, y) \in f_1$.

b) $(x, y) \in f$ and $x \in C \Leftrightarrow (x, y) \in f_2$.

If $(x, y) \in f_1$, then $x \in B$ because dom $f_1 = B$, and $(x, y) \in f$ because $f = f_1 \cup f_2$. Conversely, suppose $(x, y) \in f$ and $x \in B$: $(x, y) \in f$ implies that $(x, y) \in f_1$ or $(x, y) \in f_2$; if $(x, y) \in f_2$, then $x \in C$ (because dom $f_2 = C$), which is impossible because $x \in B$ and $B \cap C = \varnothing$; thus, $(x, y) \in f_1$. This proves (**a**); the proof of (**b**)is analogous. Next, we will prove that

c) dom $f = B \cup C$ and ran $f \subseteq A$.

Indeed, by 1.43,

$$\operatorname{dom}(f_1 \cup f_2) = \operatorname{dom}\ f_1 \cup \operatorname{dom}\ f_2 = B \cup C,$$

and

$$\operatorname{ran}(f_1 \cup f_2) = \operatorname{ran}\ f_1 \cup \operatorname{ran}\ f_2 \subseteq A.$$

Our next step will be to prove that

d) f satisfies Condition F2.

Suppose $(x, y_1) \in f$ and $(x, y_2) \in f$; now $x \in$ dom f ,so by(**c**), $x \in B$ or $x \in C$. If $x \in B$, then, by (**a**), $(x, y_1) \in f_1$ and $(x, y_2) \in f_1$, so by 2.1, $y_1 = y_2$; if $x \in C$, then, by (**b**), $(x, y_1) \in f_2$ and $(x, y_2) \in f_2$, so by 2.1, $y_1 = y_2$; this proves (**d**). From (**c**), (**d**), and Theorem 2.3, we conclude that $f : B \cup C \to A$ is a function.

By (**a**) and 2.14, $(x, y) \in f_1 \Leftrightarrow (x, y) \in f_{[B]}$, that is, $f_1 = f_{[B]}$; analogously, $f_2 = f_{[C]}$.

Finally, (**a**) states that

$$[y = f(x) \text{ and } x \in B] \Leftrightarrow y = f_1(x)$$

and (**b**) states that

$$[y = f(x) \text{ and } x \in C] \Leftrightarrow y = f_2(x);$$

thus (iii) holds. ■

EXERCISES 2.2

1. Prove that the functions introduced in 2.10 through 2.14 qualify as functions under Definition 2.1.
2. Prove that if $f : A \to B$ is an injective function and $C \subseteq A$, then $f_{[C]} : C \to B$ is an injective function.
3. Let A be a class and let $f = \{(x, (x, x)) : x \in A\}$. Show that f is a bijective function from A to I_A.
4. Let $f : A \to B$ and $g : A \to B$ be functions. Prove that if $f \subseteq g$ then $f = g$.
5. Let $f : A \to B$ and $g : C \to D$ be functions. The *product* of f and g is the function defined as follows:

$$[f \cdot g](x, y) = (f(x), g(y)) \quad \text{for every } (x, y) \in A \times C.$$

Prove that $f \bullet g$ is a function from $A \times C$ to $B \times D$. Prove that if f and g are injective, then $f \bullet g$ is injective, and if f and g are surjective, then $f \bullet g$ is surjective. Prove that ran$[f \bullet g]$= (ran f) × (ran g).

6. If $f : B \cup C \to A$ is a function, prove that $f = f_{[B]} \cup f_{[C]}$.
7. Let $f_1 : A \to B$ and $f_2 : C \to D$ be bijective function, where $A \cap C = \emptyset$ and $B \cap D = \emptyset$. Let $f = f_1 \cup f_2$; prove that $f : A \cup C \to B \cup D$ is a bijective function.
8. Let $f : B \to A$ and $g : C \to A$ be functions, and suppose that $f_{[B \cap C]} = g_{[B \cap C]}$. If $h = f \cup g$, prove that h: $B \cup C \to A$ is a function, $f = h_{[B]}$ and $g = h_{[C]}$.
9. Let $f : A \to B$ be a function; prove that f is in one-to-one correspondence with A.

By a *functional graph* we mean a graph which satisfies Condition F2. Thus G is a functional graph if and only if

$$(x, y_1) \in G \quad \text{and} \quad (x, y_2) \in G \qquad \text{implies} \quad y_1 = y_2.$$

10. If G is a functional graph, show that every subclass of G is a functional graph.
11. Let G be a graph. Prove that G is a functional graph if and only if for arbitrary graphs H and J .

$$(H \cap J) \circ G = (H \circ G) \cap (J \circ G).$$

12. Let G be a functional graph. Prove that G is injective if and only if for arbitrary graphs J and H .

$$G \circ (H \cap J) = (G \circ H) \cap (G \circ J).$$

3 PROPERTIES OF COMPOSITE FUNCTIONS AND INVERSE FUNCTIONS

The following theorems express a few basic properties of functions.

2.17 Theorem If $f : A \to B$ and $g : B \to C$ are functions, then $g \circ f : A \to C$ is a function.

Proof

i) By 1.38, dom $g \circ f$= dom $f = A$; by 1.37(iv), ran $g \circ f \subseteq$ ran $g \subseteq C$.

ii) Suppose $(x, z_1) \in g \circ f$and $(x, z_2) \in g \circ f$; by 1.34, $\exists y_1\ (x, y_1) \in f$ and $(y_1,z_1) \in g$ and $\exists y_2\ (x, y_2) \in f$ and $(y_2,z_2) \in g$. From $(x, y_1) \in f$ and $(x, y_2) \in f$ we conclude, by 2.1, that $y_1 = y_2$; thus $(y_1,z_1) \in g$ and $(y_1,z_2) \in g$. It follows by F2 (applied to g) that $z_1 = z_2$; thus, $g \circ f$satisfies F2.

From (i), (ii), and 2.3, we conclude that $g \circ f : A \to C$ is a function. ■

By 1.34 $(x, y) \in g \circ f$if and only if for some element z, $(x, z) \in f$ and $(z, y) \in g$. Thus, $x \xmapsto{g \circ f} y$ if and only if for some z, $x \xmapsto{f} z$ and $z \xmapsto{g} y$. (The reader may, if he wishes, picture this statement as in Fig. 5.) This is the same as saying that $y =[g \circ f](x)$ if and only if for some z, $z = f(x)$ and $y = g(z)$. Thus

2.18
$$[g \circ f](x) = g(f(x)).$$

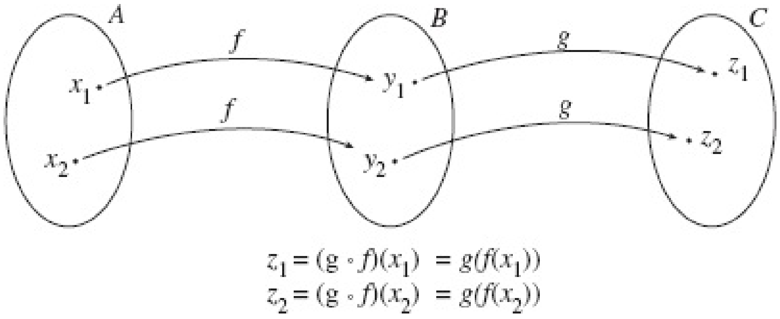

A
B
C
f
f
g
g
x_1
x_2
y_1
y_2
z_1
z_2
$z_1 = (g \circ f)(x_1) = g(f(x_1))$
$z_2 = (g \circ f)(x_2) = g(f(x_2))$

Fig.5

2.19 Definition A function $f : A \to B$ is said to be *invertible* if $f^{-1} : B \to A$ is a function.

Let $f : A \to B$ be an invertible function; by 1.33, $(x, y) \in f$ if and only if $(y, x) \in f^{-1}$. Thus $x \xmapsto{f} y$ if and only if $y \xmapsto{f^{-1}} x$. (The reader may, if he wishes, picture this statement as in Fig. 6.) Thus

2.20 $y = f(x)$ **if and only if** $x = f^{-1}(y)$.

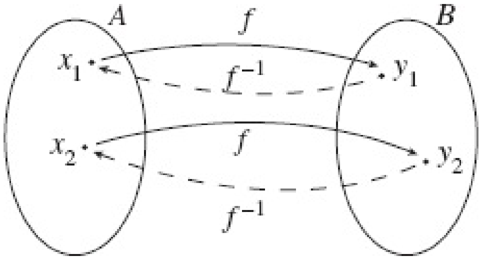


Fig.6

The next two theorems give a necessary and sufficient condition for a function to be invertible.

2.21 Theorem If $f : A \to B$ is a bijective function, then $f^{-1} : B \to A$ is a bijective function.

Proof. By 2.3, dom $f = A$, and by 2.7, ran $f = B$; thus, by 1.37, dom $f^{-1} = B$ and ran $f^{-1} = A$. Now we will prove that f^{-1} satisfies F2:

$$\begin{aligned} (y, x_1) \in f^{-1} \text{ and } (y, x_2) \in f^{-1} &\Rightarrow (x_1, y) \in f \text{ and } (x_2, y) \in f && \text{by 1.33} \\ &\Rightarrow x_1 = x_2 && \text{by 2.6.} \end{aligned}$$

Thus, by Theorem 2.3, $f^{-1} : B \to A$ is a function.

Next, we will prove that f^{-1} satisfies INJ:

$$\begin{aligned} (y_1, x) \in f^{-1} \text{ and } (y_2, x) \in f^{-1} &\Rightarrow (x, y_1) \in f \text{ and } (x, y_2) \in f && \text{by 1.33} \\ &\Rightarrow y_1 = y_2 && \text{by 2.1.} \end{aligned}$$

Finally, f^{-1} satisfies **SURJ** because (see above) ran $f^{-1} = A$. ■

2.22 Theorem If $f : A \to B$ is invertible, then $f : A \to B$ is bijective.

Proof. Let $f : A \to B$ be invertible; that is, let $f^{-1} : B \to A$ be a function. By 2.3, dom $f^{-1} = B$, so by 1.37(ii), ran $f = B$; thus, $f : A \to B$ is surjective. Now

$$\begin{aligned} (x_1,y)\in f \text{ and } (x_2,y)\in f &\Rightarrow (y,x_1)\in f^{-1} \text{ and } (y,x_2)\in f^{-1} && \text{by 1.33} \\ &\Rightarrow x_1 = x_2 && \text{by F2 (applied to } f^{-1}\text{).} \end{aligned}$$

Thus, $f : A \to B$ is injective. ■

Theorem 2.21 and 2.22 may be summarized as follows:

$f : A \to B$ is invertible if and only if it is bijective; furthermore, if $f : A \to B$ is invertible, then $f^{-1} : B \to A$ is bijective.

The next two theorems give another useful characterization of invertible functions.

2.23 Theorem Let $f : A \to B$ be an invertible function. Then

i) $f^{-1} \circ f = I_A$, and ii) $f \circ f^{-1} = I_B$.

Proof

i) Let $x \in A$ and let $y = f(x)$; then by 2.20, $x = f^{-1}(y)$. Thus

$$[f^{-1} \circ f](x) = f^{-1}[f(x)] = f^{-1}(y) = x = I_A(x);$$

this holds for every $x \in A$, so by 2.5, $f^{-1} \circ f = I_A$.

ii) The proof is analogous to (i), and is left as an exercise. ■

2.24 Theorem Let $f : A \to B$ and $g : B \to A$ be functions. If $g \circ f = I_A$ and $f \circ g = I_B$, then $f : A \to B$ is bijective (hence invertible), and $g = f^{-1}$.

Proof

i) First, we will prove that $f : A \to B$ is injective.

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow g(f(x_1)) = g(f(x_2)) && \text{by F2° (applied to } g\text{)} \\ &\Rightarrow [g \circ f](x_1) = [g \circ f](x_2) && \text{by 2.18} \\ &\Rightarrow x_1 = x_2 && \text{because } g \circ f = I_A. \end{aligned}$$

ii) Next, we will prove that $f : A \to B$ is surjective. If $y \in B$ then $y = I_B(y) = [f \circ g](y) = f(g(y))$; in other words, if y is any element of B, then $y = f(x)$, where $x = g(y) \in A$.

iii) Finally, we will prove that $g = f^{-1}$. To begin with,

$$\begin{aligned} x = g(y) &\Rightarrow f(x) = f(g(y)) = [f \circ g](y) = I_B(y) = y \\ &\Rightarrow x = f^{-1}(y); \end{aligned}$$

conversely,

$$\begin{aligned} x = f^{-1}(y) &\Rightarrow y = f(x) \\ &\Rightarrow g(y) = g(f(x)) = [g \circ f](x) = I_A(x) = x. \end{aligned}$$

Thus, $\forall y \in B$, $x = f^{-1}(y)$ iff $x = g(y)$; that is, $f^{-1}(y) = g(y)$; it follows (by 2.5) that $f^{-1} = g$. ■

Theorem 2.23 and 2.24 may be summarized as follows:

$f : A \to B$ is invertible if and only if there exists a function $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$. The function g, if it exists, is the inverse of f.

Our next theorem gives an important characterization of injective functions.

2.25 Theorem Let $f : A \to B$ be a function; $f : A \to B$ is injective if and only if there exists a function $g : B \to A$ such that $g \circ f = I_A$.

Proof

i) Suppose there exists a function $g : B \to A$ such that $g \circ f = I_A$. To prove that $f : A \to B$ is injective, we repeat part (i) of the proof of 2.24.

ii) Conversely, suppose that $f : A \to B$ is injective; let $C = \operatorname{ran} f$. By 2.4, $f : A \to C$ is a function; $f : A \to C$ is surjective (because $C = \operatorname{ran} f$), hence it is bijective; thus $f^{-1} : C \to A$ is a function. If a is some fixed element of A, let $K_a : (B - C) \to A$ be the constant function (see 2.11) which maps every element of $B - C$ onto a. If $g = f^{-1} \cup K_a$, then, by 2.16(i), $g : B \to A$ is a function (see Fig. 7). Finally, if $x \in A$, let $y = f(x)$; then

$$\begin{aligned} [g \circ f](x) = g(f(x)) &= g(y) && \text{because } y = f(x) \\ &= f^{-1}(y) && \text{by 2.16(iii)} \\ &= x && \text{because } x = f^{-1}(y) \text{ by 2.20.} \end{aligned}$$

Thus $\forall x \in A$, $[g \circ f](x) = I_A(x)$; it follows by 2.5 that $g \circ f = I_A$. ■

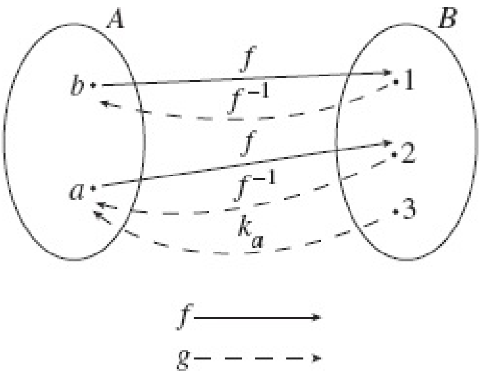


Fig.7

In Chapters 5 we will prove a companion theorem to 2.25, which will state the following: $f : A \to B$ is surjective if and only if there exists a function $g : B \to A$ such that $[f \circ g] = I_B$. Theorem 2.25 and its companion are often paraphrased as follows.

Let $f : A \to B$ be a function; $f : A \to B$ is injective if and only if it has a "left inverse" and surjective if and only if it has a "right inverse".

2.26 Theorem Suppose $f : A \to B$, $g : B \to C$, and $g \circ f : A \to C$ are functions.

i) If f and g are injective, then $g \circ f$ is injective.

ii) If f and g are surjective, then $g \circ f$ is surjective.

iii) If f and g are bijective, then $g \circ f$ is bijective.

Proof

i) Suppose that f and g both satisfy INJ°: then

$$g[f(x_1)] = g[f(x_2)] \Rightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2;$$

thus $g \circ f$ satisfies INJ°.

ii) Suppose that f and g both satisfy **SURJ**: if $z \in C$, then $\exists y \in B$ $z = g(y)$; since $y \in B$, $\exists x \in A$ $y = f(x)$; thus $z = g(f(x)) = [g \circ f](x)$. Consequently, $g \circ f$ satisfies **SURJ**.

iii) This follows immediately from (i) and (ii). ■

It follows from 2.26(iii) that

the composite of two invertible functions is invertible.

Furthermore, by 1.35(iii), $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

EXERCISES 2.3

1. Let $f : A \to B$ be a function. Prove that $I_B \circ f = f$ and $f \circ I_A = f$.
2. Suppose $f : A \to B$ and $g : B \to C$ are functions. Prove that if $g \circ f$ is injective, then f is injective; prove that if $g \circ f$ is surjective, then g is surjective. Conclude that if $g \circ f$ is bijective, then f is

injective and g is surjective.

3. Give an example to show that the converse of the last statement of Exercise 2 does not hold.
4. Let $f : A \to B$ and $g : B \to A$ be functions. Suppose that $y = f(x)$ if and only if $x = g(y)$. Prove that f is invertible and $g = f^{-1}$.
5. Let $g : B \to C$ and $h: B \to C$ be functions. Suppose that $g \circ f = h \circ f$ for every function $f : A \to B$. Prove that $g = h$.
6. Suppose $g : A \to B$ and $h: A \to B$ are functions. Let C be a set with more than one element; suppose that $f \circ g = f \circ h$ for every function $f : B \to C$. Prove that $g = h$.
7. Let $f : B \to C$ be a function. Prove that f is injective if and only if, for every pair of functions $g : A \to B$ and $h: A \to B$, $f \circ g = f \circ h \Rightarrow g = h$.
8. Let $f : A \to B$ be a function. Prove that f is surjective if and only if, for every pair of functions $g : B \to C$ and $h: B \to C$, $g \circ f = h \circ f \Rightarrow g = h$.
9. Let $f : A \to C$ and $g : A \to B$ be functions. Prove that there exists a function $h: B \to C$ such that $f = h \circ g$ if and only if $\forall x, y \in A$,

$$g(x) = g(y) \Rightarrow f(x) = f(y).$$

   Prove that h is unique.

10. Let $f : C \to A$ and $g : B \to A$ be functions, and suppose that g is bijective. Prove that there exists $h: C \to B$ such that $f = g \circ h$ if and only if ran $f \subseteq$ ran g. Prove that h is unique.
11. Let $f : A \to B$ be a function, and let $C \subseteq A$. Prove that $f_{[C]} = f \circ E_C$, where E_C is the inclusion function of C in A (2.12).

4 DIRECT IMAGES AND INVERSE IMAGES UNDER FUNCTIONS

2.27 Definition Let $f : A \to B$ be a function; if C is any subclass of A, the direct image of C under f , which we write $\bar{f}(C)$, is the following subclass of B:

$$\bar{f}(C) = \{y \in B : \exists x \in C \ni y = f(x)\}.$$

That is, $\bar{f}(C)$ is the class of all the images of elements in C.

2.28 Definition Let $f : A \to B$ be a function; if D is any subclass of B, the inverse image of D under f , which we write $\check{f}$(D), is the following subclass of A:

$$\check{f}(D) = \{x \in A : f(x) \in D\}.$$

That is, $\check{f}$(D) is the class of all the pre-images of elements in D.

If $\{a\}$ and $\{b\}$ are singletons, we will write $\bar{f}$(a) for $\bar{f}(\{a\})$ and $\check{f}$(b) for $\check{f}(\{b\})$.

2.29 Theorem Let $f : A \to B$ be a function.

i) if $C \subseteq A$ and $D \subseteq A$, then $C = D \Rightarrow \bar{f}(C) = \bar{f}$(D).

ii) if $C \subseteq B$ and $D \subseteq B$, then $C = D \Rightarrow \check{f}$(C) = $\check{f}$(D).

Proof

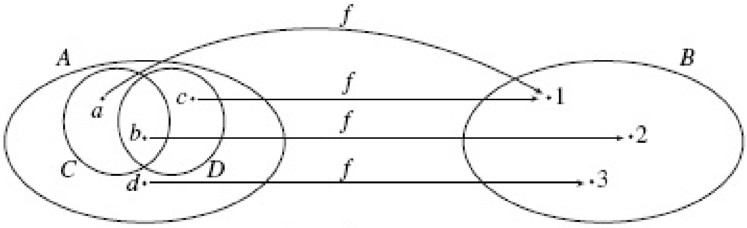
i) Suppose $C = D$; then

$$\begin{aligned} y \in \bar{f}(C) &\Leftrightarrow \exists x \in C \ni y = f(x) && \text{by 2.27} \\ &\Leftrightarrow \exists x \in D \ni y = f(x) && \text{because } C = D \\ &\Leftrightarrow y \in \bar{f}(D) && \text{by 2.27.} \end{aligned}$$

ii) Suppose $C = D$; then

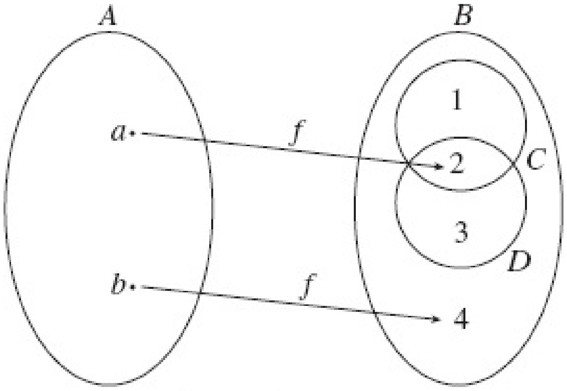
$$\begin{aligned} x \in \breve{f}(C) &\Leftrightarrow f(x) \in C && \text{by 2.28} \\ &\Leftrightarrow f(x) \in D && \text{because } C = D \\ &\Leftrightarrow x \in \breve{f}(D) && \text{by 2.28. } \blacksquare \end{aligned}$$

Caution. $\bar{f}(C)$= $\bar{f}$(D)does not always imply that $C = D$; for a simple counterexample, see Fig. 8. Similarly, $\breve{f}$(C) = $\breve{f}$(D) does not always imply $C = D$; for a counterexample, the reader should look at Fig. 9. (*However, see Exercise* 3, Exercise Set 2.4.)



$\bar{f}(C) = \bar{f}(D) = \{1, 2\}.$

Fig.8



$\breve{f}(C) = \breve{f}(D) = \{a\}.$

Fig.9

2.30 Theorem Let A and B be sets and let $f : A \to B$ be a function; then

i) $\bar{f} : \mathscr{P}(A) \to \mathscr{P}(B)$ is a function.

ii) $\breve{f} : \mathscr{P}(B) \to \mathscr{P}(A)$ is a function.

Proof

i) By 2.27, it is easy to see that dom $\bar{f}$= $\mathscr{P}(A)$and ran $\bar{f}\subseteq$ $\mathscr{P}(B)$. Theorem 2.29(i) states that $\bar{f}$satisfies Condition F2; thus by 2.3, $\bar{f}$: $\mathscr{P}(A) \to$ $\mathscr{P}(B)$ is a function.

ii) Analogously, $\check{f}: \mathcal{P}(B) \to \mathcal{P}(A)$ is a function. ■

2.31 Theorem Let $f : A \to B$ be a function, let $\{C_i\}_{i\in I}$ be a family of subclasses of A, and let $\{D_i\}_{i\in I}$ be a family of subclasses of B. Then

i) $\bar{f}(\bigcup_{i\in I} C_i) = \bigcup_{i\in I} \bar{f}(C_i)$.

ii) $\check{f}(\bigcup_{i\in I} D_i) = \bigcup_{i\in I} \check{f}(D_i)$.

iii) $\check{f}(\bigcap_{i\in I} D_i) = \bigcap_{i\in I} \check{f}(D_i)$.

Proof

$$\begin{aligned}
\text{i)}\ \ y \in \bar{f}(\bigcup_{i\in I} C_i) &\Leftrightarrow \exists x \in \bigcup_{i\in I} C_i \ni y = f(x) && \text{by 2.27}\\
&\Leftrightarrow \text{for some } j \in I, \exists x \in C_j \ni y = f(x) && \text{by 1.39}\\
&\Leftrightarrow \text{for some } j \in I, y \in \bar{f}(C_j) && \text{by 2.27}\\
&\Leftrightarrow y \in \bigcup_{i\in I} \bar{f}(C_i) && \text{by 1.39.}
\end{aligned}$$

$$\begin{aligned}
\text{ii)}\ \ x \in \check{f}(\bigcup_{i\in I} D_i) &\Leftrightarrow f(x) \in \bigcup_{i\in I} D_i && \text{by 2.28}\\
&\Leftrightarrow \text{for some } j \in I, f(x) \in D_j && \text{by 1.39}\\
&\Leftrightarrow \text{for some } j \in I, x \in \check{f}(D_j) && \text{by 2.28}\\
&\Leftrightarrow x \in \bigcup_{i\in I} \check{f}(D_i) && \text{by 1.39.}
\end{aligned}$$

iii) The proof is left as an exercise for the reader. ■

Caution. It is important to note that there is no counterpart of Theorem 2.31(iii) for $\bar{f}$; more precisely, we have

$$\bar{f}(\bigcap_{i\in I} C_i) \subseteq \bigcap_{i\in I} \bar{f}(C_i),$$

but we do not have inclusion the other way. For a simple counterexample, see Fig. 8, where $\bar{f}(C \cap D) = \bar{f}(b) = \{2\} \neq \{1, 2\} = \bar{f}(C) \cap \bar{f}(D)$. For this reason, a variety of theorems which are true for inverse images of sets fail to hold for direct images of sets.

EXERCISES 2.4

1. Suppose that $f : A \to B$ is a function, $C \subseteq A$ and $D \subseteq B$.

a) Prove that $C \subseteq \check{f}[\bar{f}(C)]$. b) Prove that $\bar{f}[\check{f}(D)] \subseteq D$.

2. Suppose that $f : A \to B$ is a function, $C \subseteq A$ and $D \subseteq B$.

   a) If f is injective, prove that $C = \check{f}[\ \bar{f}(C)]$.

   b) If f is surjective, prove that $D = \bar{f}[\ \check{f}(D)]$.

3. Let $f : A \to B$ be a function. Prove the following.

   a) Suppose $C \subseteq A$ and $D \subseteq A$;if f is injective, then $\bar{f}(C) = \bar{f}(D) \Rightarrow C = D$.

   b) Suppose $C \subseteq B$ and $D \subseteq B$; if f is surjective, then $\check{f}(C) = \check{f}(D) \Rightarrow C = D$.

   [*Hint*: Use the result of Exercise 2.]

4. Let $f : A \to B$ be a function. Prove the following:

   a) If f is injective, then $\check{f} \circ \bar{f}$ is bijective. [*Hint*: Use the result of Exercise 2(a).]

   b) If f is surjective, then $\bar{f} \circ \check{f}$ is bijective. [*Hint*: Use the result of Exercise 2(b).]

5. Suppose that $f : A \to B$ is a function; let $C \subseteq A$.

   a) Prove that $\bar{f}\{\ \check{f}[\ \bar{f}(C)]\} = \bar{f}(C)$.

   b) Use the result of (a) to prove that $\bar{f} \circ \check{f} \circ \bar{f} = \bar{f}$.

6. Let $f : A \to B$ be a function. Prove the following:

   a) If f is injective, then $\bar{f}$ is injective.

   b) If f is surjective, then $\bar{f}$is surjective.

   c) If f is bijective, then $\bar{f}$is bijective.

7. Let $f : A \to B$ be a function. Prove the following:

   a) If f is injective, then $\check{f}$ is surjective.

   b) If f is surjective, then $\check{f}$ is injective.

   c) If f is bijective, then $\check{f}$ is bijective.

8. Let $f : A \to B$ be a function. Prove that

$$\bar{f}(C \cap D) = \bar{f}(C) \cap \bar{f}(D)$$

   for every pair of subclasses $C \subseteq A$ and $D \subseteq A$ if and only if f is injective.

9. Suppose that $f : A \to B$ is a function, $C \subseteq B$ and $D \subseteq B$. Prove that

$$\check{f}(C - D) = \check{f}(C) - \bar{f}(D).$$

10. Let $f : A \to B$ be a function. Prove each of the following:

    a) If $C \subseteq A$ and $D \subseteq A$, then $\bar{f}(C) - \bar{f}(D) \subseteq \bar{f}(C - D)$.

    b) $\bar{f}(C) - \bar{f}(D) \subseteq \bar{f}(C - D)$ for every pair of subclasses $C \subseteq A$ and $D \subseteq A$ if and only if f is injective.

5 PRODUCT OF A FAMILY OF CLASSES

In the beginning of Chapters 1 we spoke of the union and intersection of two classes; later, we extended this notion by defining the union and intersection of an arbitrary family of classes. In much the same manner, we will now extend the notion of the Cartesian product of two classes by defining the product of a family of classes.

The product of two classes A and B has been defined to be the class $A \times B$ of all ordered pairs (x, y), where $x \in A$ and $y \in B$. This definition may be extended, in a natural way, to a finite number of classes $A_1, A_2, \ldots, A_n$; we may define the product $A_1 \times A_2 \times \ldots \times A_n$ to be the class of all "ordered n-tuples" $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for each index $i = 1, 2, \ldots, n$. Now, we wish to extend this concept to the case of an indexed family of classes, $\{A_i\}_{i \in I}$, where the index class I is any class whatsoever. Evidently we cannot speak of "I -tuples" of elements, because I may be an infinite class and may fail to be ordered; therefore, we must alter our approach to the problem.

Let us take another look at the product $A_1 \times A_2 \times \ldots \times A_n$. Clearly $\{A_1, A_2, \ldots, A_n\}$ is a family whose index class is $I = \{1, 2, \ldots, n\}$. Now, *an ordered n-tuple may be regarded as a function (whose domain is I) which maps each element $i \in I$ onto an element a_i in A_i*. Indeed, if f is such a function, then f is described by the following table.

x	$f(x)$
1	a_1
2	a_2
$\vdots$	
n	a_n

Using the table, we may construct the ordered n-tuple $(a_1, a_2, \ldots, a_n)$; conversely, if we are give the ordered n-tuple $(a_1, a_2, \ldots, a_n)$, then we may construct the table; (in fact, the ordered n-tuple is simply the table presented as a horizontal array). Thus, the function f and the ordered n-tuple $(a_1, a_2, \ldots, a_n)$ are, essentially, one and the same thing. This simple observation leads to the following definition of the product of a family of classes.

2.32 Definition Let $\{A_i\}_{i \in I}$ be an indexed family of classes; let

$$A = \bigcup_{i \in I} A_i .$$

The *product* of the classes A_i is defined to be the class

$$\prod_{i \in I} A_i = \{f : f : I \to A \text{ is a function, and } f(i) \in A_i, \forall i \in I\}.$$

2.33 Example Let $I = \{1, 2\}$, $A_1 = \{a, b\}$, and $A_2 = \{c, d\}$. By 2.32, $\prod_{i \in I} A_i$ consists of all the functions $f : \{1, 2\} \to \{a, b, c, d\}$ such that $f(1) \in A_1$ and $f(2) \in A_2$. There are four such functions, given by the following tables.

x	$f(x)$
1	a
2	c

x	$f(x)$
1	a
2	d

x	$f(x)$
1	b
2	c

x	$f(x)$
1	b
2	d

We may identify these four functions with the four ordered pairs *(a, c)*, *(a, d)*, *(b, c)*, and *(b, d)*, respectively. Thus $\prod_{i \in I} A_i$ is exactly $A_1 \times A_2$.

We adopt the following notational convention: henceforth, we will designate elements of a product $\prod_{i \in I} A_i$ by bold face letters **a**, **b**, **c**, etc.

If **a** is an element of $\prod_{i \in I} A_i$ and $j \in I$, we agree that $\mathbf{a}_j$ will have the same meaning as $\mathbf{a}(j)$; we will call $\mathbf{a}_j$ the *j-coordinate* of **a**.

Let $\{A_i\}_{i \in I}$ be an indexed family, and, for each $i \in I$, let $x_i \in A_i$. We will use the symbol $\{x_i\}_{i \in I}$ to designate the element in $\prod_{i \in I} A_i$ whose *i*-coordinate, for each $i \in I$, is x_i.

Let $A = \prod_{i \in I} A_i$; corresponding to each index $i \in I$, we define a function p_i from A to A_i by

$$p_i(\mathbf{a}) = \mathbf{a}_i, \quad \forall \mathbf{a} \in A.$$

The function p_i is called the *i-projection* of A to A_i.

2.34 Definition If A and B are arbitrary classes, the symbol B^A refers to the class of all functions from A to B.

In particular, if 2 denotes a class of two elements, then 2^A denotes the class of all functions from A to 2. The following is an important result which will be used in a later chapter.

2.35 Theorem If A is a set, then 2^A and $\mathscr{P}(A)$ are in one-to-one correspondence.

Proof. We will show that there exists a bijective function $\gamma : \mathscr{P}(A) \to 2^A$. If $B \in \mathscr{P}(A)$, let C_B denote the characteristic function of B in A (see 2.13); C_B is an element of 2^A. We define γ by

$$\gamma(B) = C_B, \quad \forall B \in \mathscr{P}(A).$$

By the way γ is defined, it is clear that γ maps every $B \in \mathscr{P}(A)$ onto a uniquely determined element of 2^A; hence $\gamma : \mathscr{P}(A) \to 2^A$ is a function; it remains to show that γ is injective and surjective.

i) Let $B, D \in \mathscr{P}(A)$; if $\gamma(B) = \gamma(D)$, then $C_B = C_D$; hence

$$\{x \in A : C_B(x) = 0\} = \{x \in A : C_D(x) = 0\};$$

that B = D. Thus γ satisfies INJ°.

ii) If $f \in 2^A$, and if we let $B = \check{f}(0)$, then $f = C_B = \gamma(B)$. Thus γ satisfies condition **SURJ**. ■

It is easy to show that if A and B are sets, then A^B is a set (see Exercise 12, Exercise Set 2.5). Using this fact, it can easily be shown that if $\{A_i\}_{i \in I}$ is an indexed family of *sets* such that the index class I is a *set*, then $\prod_{i \in I} A_i$ is a set (see Exercise 13, Exercise Set 2.5, and Remark 2.38).

EXERCISES 2.5

1. Let A ={1, 2, 3}, B ={a, b}. Find A^B,B^A, 2^A, and $\mathscr{P}(A)$.
2. Suppose that $\{B_i\}_{i\in I}$ is a family of subclasses of A. Prove that

$$\prod_{i\in I} B_i \subseteq A^I.$$

3. Suppose that $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ are families of classes with the same index class I . Show that if $A_i \subseteq B_i$, $\forall i \in I$, then

$$\prod_{i\in I} A_i \subseteq \prod_{i\in I} B_i.$$

In the next three exercises (Exercises 4, 5 and 6), assume the following:

$$\bigcup_{i\in I} A_i = \bigcup_{j\in J} B_j = X.$$

4. Suppose that $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ are families of nonempty classes with the same index class I . Prove that if

$$\prod_{i\in I} A_i \subseteq \prod_{i\in I} B_i,$$

then $A_i \subseteq B_i$ for each index i.

5. Suppose that $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ are families of classes with the same index class I . Prove that

$$(\prod_{i\in I} A_i) \cap (\prod_{i\in I} B_i) = \prod_{i\in I} (A_i \cap B_i).$$

6. Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be families of classes. Prove the following:

$$\text{a) } (\prod_{i\in I} A_i) \cap (\prod_{j\in J} B_j) = \prod_{(i,j)\in I\times J} (A_i \cap B_j).$$
$$\text{b) } (\prod_{i\in I} A_i) \cup (\prod_{j\in J} B_j) = \prod_{(i,j)\in I\times J} (A_i \cup B_j).$$

7. Let $\{A_i\}_{i\in I}$ be a family of classes, and for each $i \in I$, let B_i be a subclass of A_i. Prove that

$$\bigcap_{i\in I} \breve{p}_i(B_i) = \prod_{i\in I} B_i.$$

8. Let $\{A_i\}_{i\in I}$ be an indexed family, and let

$$A = \prod_{i\in I} A_i.$$

If $B \subseteq A$, let $B_i = {}^{-}p_i(B)$ for each $i \in I$. Prove that $B \subseteq$

9. Prove that $A^C \cup B^C \subseteq (A \cup B)^C$.

10. Prove that $(A \cap B)^C = A^C \cap B^C$.

11. Prove that $(A - B)^C = A^C - B^C$.

12. Prove that if A and B are sets, then A^B is a set. [*Hint*: Each element of A^B is a subset of $B \times A$. Use the axioms of the last two chapters as needed.]

13. Let $\{A_i\}_{i \in I}$ be an indexed family; suppose that I is a set, that each A_i is a set, and that $\{A_i : i \in I\}$ is a set. Prove that is a set. [*Hint*: Use the results of Exercises 2 and 12.]

6 THE AXIOM OF REPLACEMENT

Axioms A3 through A7 are “set” axioms, that is, they are designed for the purpose of establishing the properties of sets. We are now in a position to introduce our last “set” axiom. This axiom is motivated by the following considerations.

We noted earlier that we are to think of a *set* as a class which is “not too large.” Now, if A and B are classes and $f : A \to B$ is a surjective function, then, in an obvious intuitive sense, B has “as many, or fewer elements than A“ (see Fig. 3). Thus if A is “not too large” and $f : A \to B$ is a surjective function, it stands to reason that B is “not too large”. These remarks lead us to state the following as an axiom.

A9. If A is a set and $f : A \to B$ is a surjective function, than B is a set.

Statement A9 is traditionally called the *axiom of replacement*; it has the following consequences.

2.36 If A is a set and A is in one-to-one correspondence with B, then B is a set.

We noted on page 47 that Ø is a set, hence by Axiom A5 {Ø, Ø} is a set, that is, {Ø} is a set. Thus, by 2.36,

2.37 *every singleton is a set.*

Since the union of two sets is a set, it follows that every doubleton is a set; similarly, every class of three elements is a set, every class of four elements is a set, and so on, through all the positive integers. Thus, in an intuitive sense, every finite class is a set.

2.38 *Remark*. Let $\{A_i\}_{i \in I}$ be an indexed family of sets, where the index class I is a set. It is clear that the function φ defined by $\varphi(i) = A_i$ is a surjective function from I to $\{A_i : i \in I\}$; thus we have

If $\{A_i\}_{i \in I}$ is an indexed family of sets and I is a set, then $\{A_i : i \in I\}$ is a set.

3

Relations

1 INTRODUCTION

Intuitively, a binary relation in a class A is a statement $R(x, y)$ which is either true or false for each ordered pair (x, y) of elements of A. For instance, the relation "*x divides y*," which we may write $D(x, y)$, is a relation in the class $\mathbb{Z}$ of the integers: $D(x, y)$ is true for every pair (x, y) of integers such that y is a multiple of x; it is false for every other pair of integers.

The *representing graph* of a relation in A is a graph $G \subseteq A \times A$ which consists of all the pairs (x, y) such that $R(x, y)$ is true. Conversely, if we are given an arbitrary graph $G \subseteq A \times A$, then G defines a relation in A, namely the relation R such that $R(x, y)$ is true if and only if $(x, y) \in G$.

Thus, as we did in the case of functions, we are able to identify relations with their representing graphs. In this way the study of relations is part of elementary set theory.

2 FUNDAMENTAL CONCEPTS AND DEFINITIONS

3.1 Definition Let A be a class; by a *relation in A* we mean an arbitrary subclass of $A \times A$.

3.2 Definition Let G be a relation in A; then

G is called *reflexive* if

$$\forall x \in A, (x, x) \in G.$$

G is called *symmetric* if

$$(x, y) \in G \Rightarrow (y, x) \in G.$$

G is called *anti-symmetric* if

$$(x, y) \in G \text{ and } (y, x) \in G \Rightarrow x = y.$$

G is called *transitive* if

$$(x, y) \in G \text{ and } (y, z) \in G \Rightarrow (x, z) \in G.$$

3.3 Definition *The diagonal graph* I_A is defined to be the class $\{(x, x) : x \in A\}$.

It is easy to see that G is reflexive if and only if $I_A \subseteq G$.

There is a variety of interesting and useful alternative ways of defining the above notions. Some are given in the next theorem.

3.4 Theorem Let G be a relation in A.

i) G is symmetric if and only if $G = G^{-1}$.

ii) G is antisymmetric if and only if $G \cap G^{-1} \subseteq I_A$.

iii) G is transitive if and only if $G \circ G \subseteq$ G.

Proof

i) Suppose G is symmetric. Then

$$(x, y) \in G \Leftrightarrow (y, x) \in G \Leftrightarrow (x, y) \in G^{-1};$$

thus $G = G^{-1}$. Conversely, suppose $G = G^{-1}$. Then

$$(x, y) \in G \Rightarrow (x, y) \in G^{-1} \Rightarrow (y, x) \in G.$$

ii) Suppose G is antisymmetric. Then

$$\begin{aligned}(x, y) \in G \cap G^{-1} &\Rightarrow (x, y) \in G \text{ and } (x, y) \in G^{-1}\\ &\Rightarrow (x, y) \in G \text{ and } (y, x) \in G\\ &\Rightarrow x = y\\ &\Rightarrow (x, y) = (x, x) \in I_A.\end{aligned}$$

Conversely, suppose that $G \cap G^{-1} \subseteq I_A$. Then

$$\begin{aligned}(x, y) \in G \text{ and } (y, x) \in G &\Rightarrow (x, y) \in G \text{ and } (x, y) \in G^{-1}\\ &\Rightarrow (x, y) \in G \cap G^{-1} \subseteq I_A\\ &\Rightarrow x = y.\end{aligned}$$

ii) Suppose G is transitive. Then

$(x, y) \in G \circ G \Rightarrow \exists z\ (x, z) \in G$ and $(z, y) \in G \Rightarrow (x, y) \in G$. Thus $G \circ G$.

Conversely, suppose $G \circ G$: Then $(x, y) \in G$ and $(y, z) \in G \Rightarrow (x, z) \in G \circ G \subseteq G$. ■

3.5 Definition A relation is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

A relation is called an *order relation* if it is reflexive, antisymmetric, and transitive.

3.6 Definition Let G be a relation in A.

G is called *irreflexive* if

$$\forall x \in A, \quad (x, x) \notin G.$$

G is called *asymmetric* if

$$(x, y) \in G \Rightarrow (y, x) \notin G.$$

G is called *intransitive* if

$$(x, y) \in G \text{ and } (y, z) \in G \Rightarrow (x, z) \notin G.$$

Examples Let $\mathbb{Z}$ designate the set of the integers; the equality relation in $\mathbb{Z}$ is reflexive, symmetric, and transitive; hence, it is an equivalence relation. The relation $\leqslant$ ("less than or equal to") is reflexive, antisymmetric, and transitive; hence it is an order relation. The relation < ("strictly less than") is not an order relation: it is irreflexive, asymmetric, and transitive; such a relation is called a relation of *strict order*.

EXERCISES 3.2

1. Each of the following describes a relation in the set Z of the integers. State, for each one, whether it has any of the following properties: reflexive, symmetric, antisymmetric, transitive, irreflexive, asymmetric, intransitive. Determine whether it is an equivalence relation, an order relation, or neither. Prove your answer in each case.
   a) $G = \{(x, y) : x + y < 3\}$.
   b) $G = \{(x, y) : x$ divides $y\}$.
   c) $G = \{(x, y) : x$ and y are relatively prime$\}$.
   d) $G = \{(x, y) : x + y$ is an even number$\}$.
   e) $G = \{(x, y) : x = y$ or $x =-y\}$.
   f) $G = \{(x, y) : x + y$ is even and x is a multiple of $y\}$.
   g) $G = \{(x, y) : y = x + 1\}$.
2. Let *G* be a relation in *A*; prove each of the following:
   a) *G* is irreflexive if and only if $G \cap I = \emptyset$.
   b) *G* is asymmetric if and only if $G \cap G^{-1} = \emptyset$.
   c) *G* is intransitive if and only if $(G \circ G) \cap G = \emptyset$.
3. Show that if is an equivalence relation in *A*, then $G \circ G = G$.
4. Let $\{G_i\}_{i\in I}$ be an indexed family of equivalence relations in *A*. Show that $\bigcap_{i\in I} G_i$is an equivalence relation in *A*.
5. Let $\{G_i\}_{i\in I}$ be an indexed family of order relations in *A*. Show that $\bigcap_{i\in I} G_i$ order relation in *A*.
6. Let *H* be a reflexive relation in *A*. Prove that for any relation *G* in *A*, $G \subseteq H \circ G$ and $G \subseteq G \circ H$.
7. Let *G* and *H* be relations in *A*; suppose that *G* is reflexive and *H* is reflexive and transitive. Show that $G \subseteq H$ if and only if $G \circ H = H$. (In particular, this holds if *G* and *H* are equivalence relations.)
8. Show that the inverse of an order relation in *A* is an order relation in *A*.
9. Let *G* be a relation in *A*. Show that *G* is an order relation if and only if $G \cap G^{-1} = I_A$ and $G \circ G = G$.
10. Let *G* and *H* be equivalence relations in *A*. Show that $G \circ H$is an equivalence relation in *A* if and only if $G \circ H = H \circ G$.
11. Let *G* and *H* be equivalence relations in *A*. Prove that $G \cup H$ is an equivalence relation in *A* if and only if $G \circ H \subseteq G \cup H$ and $H \circ G \subseteq G \cup H$.
12. Let *G* be an equivalence relation in *A*. Prove that if *H* and *J* are reflexive relations in *A*, then $G \subseteq H$ and $G \subseteq J \Rightarrow G \subseteq H \circ J$.

3 EQUIVALENCE RELATIONS AND PARTITIONS

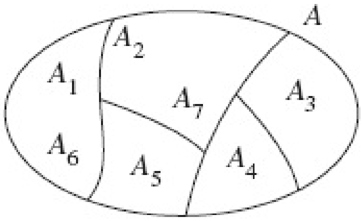
In the remainder of this chapter we will concern ourselves with equivalence relations in *sets*. The concepts we are about to introduce arise naturally in terms of sets, but cannot be extended to proper classes; to understand why not, the reader should review our discussion in Section 7 of Chapter 1. Briefly, if A is a set and $P(X)$ is a property, then by 1.52 it is legitimate to form the set of all the subsets $X \subseteq A$ which satisfy $P(X)$. However, if A were an arbitrary class, it would not be permissible to form the "class of all subclasses of A which satisfy $P(X)$." This restriction compels us to confine the following discussion to sets. Intuitively, this should not disturb the reader too much, for a set is almost the same thing as a class: a set is any class except an "excessively large" one.

3.7 Definition Let A be a set; by a *partition* of A we mean a family $\{A_i\}_{i \in I}$ of nonempty subsets of A with the following properties:

P1. $\forall i, j \in I$, $A_i \cap A_j = \emptyset$ or $A_i = A_j$.

P2. $A = \bigcup_{i \in I} A_i .$

Intuitively, a partition is a family of subsets of A which are disjoint from one another, and whose union is all of A (Fig. 1). The subsets are called the *members* of the partition. It is customary to allow a given member of the partition to be designated by more than one index; that is, we may have $A_i = A_j$, where $i \neq j$. Hence the condition that two *distinct* members be disjoint is correctly expressed by P1.



$\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ is a partition of A. Note that $A_1 = A_6$ and $A_2 = A_7$.

Fig.1

Property P1 states that any two members A_i and A_j are either disjoint or equal; that is, they have either no elements in common or all their elements in common; in other words, if they have so much as one element in common, they have all their elements in common. Thus, P1 may also be stated as follows:

P1°. If $\exists x \in A_i \cap A_j$, then $A_i = A_j$.

P2 may be replaced by the simpler condition

P2′. $A \subseteq \bigcup_{i \in I} A_i$

For, independently of Condition P2, we are given that each A_i is a subset of A; hence, by 1,40(i),

$\bigcup_{i\in I} A_i \subseteq A$. Consequently, it is sufficient to state P2′ in order to have $A \subseteq \bigcup_{i\in I} A_i$. It is convenient to write P2′ in the form.

P2°. If $x \in A$, then $x \in A_i$ for some $i \in I$.

Briefly, then, a partition of A is a family $\{A_i\}_{i\in I}$ of nonempty subsets of A such that

P1°. If $\exists x \in A_i \cap A_j$ then $A_i = A_j$ and

P2°. If $x \in A$, then $x \in A_i$ for some $i \in I$.

Examples of partitions are given in the exercises which follow this section.

The results which follow state the connection between equivalence relations in A and partitions of A. They are of great importance in many branches of mathematics.

Let G be an equivalence relation in A; we will sometimes write $x \underset{G}{\sim} y$ instead of $(x, y) \in G$, and say that "*x is equivalent to y modulo G*;" when there is no danger of ambiguity, we will write simply $x \sim y$ and say that "*x is equivalent to y*." Note that since G is an equivalence relation in A,we have

i) $x \sim x, \forall x \in A$.

ii) $x \sim y \Rightarrow y \sim x$.

iii) $x \sim y$ and $y \sim z \Rightarrow x \sim z$.

3.8 Definition Let A be a set and let G be an equivalence relation in A. If $x \in A$, then the *equivalence class of x modulo G* is the set G_x defined as follows:

$$G_x = \{y \in A : (y, x) \in G\} = \{y \in A : y \underset{G}{\sim} x\}.$$

In other words, G_x is the set of all the element of A which are equivalent to x. In the mathematical literature, G_x is also denoted by the symbols A_x, $[x]$,x/G.

3.9 Lemma Let G be an equivalence relation in A. Then

$$x \sim y \quad \text{if and only if} \quad G_x = G_y.$$

Proof

i) Suppose $x \sim y$;we have

$$\begin{aligned} z \in G_x \Rightarrow z \sim x \Rightarrow z \sim y &\qquad \text{because we assume } x \sim y \\ \Rightarrow z \in G_y. \end{aligned}$$

We have shown that $G_x \subseteq G_y$; analogously, $G_y \subseteq G_x$; hence $G_x = G_y$.

ii) Suppose $G_x = G_y$; by the reflexive property, $x \sim x$,so $x \in G_x$;but $G_x = G_y$; hence $x \in G_y$, that is, $x \sim y$. ■

3.10 Theorem Let A be a set, let G be an equivalence relation in A, and let $\{G_x\}_{x\in A}$ be the family of all the equivalence classes modulo G. Then

$$\{G_x\}_{x\in A} \text{ is a partition of } A.$$

Proof. By definition, each G_x is a subset of A; it is nonempty because $x \sim x$, hence $x \in G_x$. It remains to prove that P1°and P2° hold.

(**P1°**) $z \in G_x \cap G_y \Rightarrow z \in G_x$ and $z \in G_y \Rightarrow z \sim x$ and $z \sim y \Rightarrow x \sim z$ and $z \sim y \Rightarrow x \sim y \Rightarrow G_x = G_y$ (the last implication follows by 3.9).

(**P2°**) If $x \in A$, then by the reflexive property $x \sim x$; hence $x \in G_x$. ■

If G is an equivalence relation in A, and $\{G_x\}_{x\in A}$ is the family of all the equivalence classes modulo G, then $\{G_x\}_{x\in A}$ is referred to as the *partition induced by* G, or the *partition corresponding to* G.

Theorem 3.10 has an important converse, which follows.

3.11 Theorem Let A be a set, let $\{A_i\}_{i\in I}$ be a partition of A, and let G be the set of all pairs (x, y) of elements of A such that x and y are in the same member of the partition; that is,

$$G = \{(x, y) : x \in A_i \text{ and } y \in A_i \text{ for some } i \in I\}.$$

Then G is an equivalence relation in A, and $\{A_i\}_{i\in I}$ is the partition induced by G. G is called the *equivalence relation corresponding to* $\{A_i\}_{i\in I}$.

Proof

G is reflexive: $x \in A \Rightarrow x \in A_i$ for some $i \in I \Rightarrow x \in A_i$ and $x \in A_i \Rightarrow (x, x) \in G$.
G is symmetric : $(x, y) \in G \Rightarrow x \in A_i$ and $y \in A_i \Rightarrow y \in A_i$ and $x \in A_i \Rightarrow (y, x) \in G$.
G is transitive: $(x, y) \in G$ and $(y, z) \in G \Rightarrow x \in A_i$ and $y \in A_i$ and $y \in A_j$ and $z \in A_j \Rightarrow A_i = A_j$ (because $y \in A_i \cap A_j$) $\Rightarrow x \in A_i$ and $z \in A_i \Rightarrow (x, z) \in G$.
Finally, each A_i is an equivalence class modulo G; for suppose $x \in A_i$; then $y \in A_i \Leftrightarrow (y, x) \in G \Leftrightarrow y \in G_x$; thus $A_i = G_x$. ■

The last two theorems make it clear that every equivalence relation in A corresponds uniquely to a partition of A, and conversely. Once again: if we are given a partition of A, the *corresponding equivalence relation* is the relation which calls elements x and y “equivalent” if they are in the same member of the partition. Looking at the other side of the coin, if we are given an equivalence relation in A, the *corresponding partition* is the one which puts elements x and y in the same member of the partition iff they are equivalent. The reader should note that G is the equivalence relation corresponding to $\{A_i\}_{i\in I}$ if and only if $\{A_i\}_{i\in I}$ is the partition corresponding to G.

3.12 Example Let $A =\{a, b, c, d, e\}$; let $A_1 =\{a, b\}$, $A_2 =\{c, d\}$ and $A_3 =\{e\}$. Let $G =\{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (b, a), (c, d), (d, c)\}$. It is easy to see that $\{A_1,A_2,A_3\}$ is a partition of A (see Fig. 2), and that G is an equivalence relation in A; G is the equivalence relation corresponding to $\{A_1,A_2,A_3\}$, and $\{A_1,A_2,A_3\}$ is the partition corresponding to G. It should be noted that $A_1 = G_a = G_b$, $A_2 = G_c = G_d$, and $A_3 = G_e$.

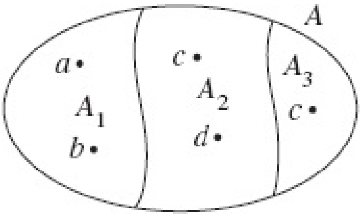


Fig.2

If G is an equivalence relation in set A, then the set of equivalence classes modulo G is called the *quotient set of A by G*, and is customarily denoted by A/G. Thus, in the preceding example, A/G is the set of three elements $\{G_a,G_c,G_e\}$. The concept of a quotient set plays a vital role in many parts of advanced mathematics.

EXERCISES 3.3

1. Let $\mathbb{Z}$ be the set of the integers. For each integer n, let $B_n = \{m \in \mathbb{Z} : \exists q\ m = n + 5q\}$. Prove that $\{B_n\}_{n\in\mathbb{Z}}$ is a partition of $\mathbb{Z}$.
2. Let $\mathbb{R}$ be the set of the real numbers. In each of the following, prove that $\{B_r\}_{r\in\mathbb{R}}$ is a partition of $\mathbb{R} \times \mathbb{R}$. Describe geometrically the members of this partition. Find the equivalence relation corresponding to each partition.

   a) $B_r = \{(x, y) : y = x + r\}$ for each $r \in \mathbb{R}$,

   b) $B_r = \{(x, y) : x^2 + y^2 = r\}$ for each $r \in \mathbb{R}$.

   [*Hint*: $y = x + r$ is the equation of a line and $x^2 + y^2 = r$ is the equation of a circle.]
3. Let $\mathbb{R}$ be the set of the real numbers. Prove that each of the following is an equivalence relation in $\mathbb{R} \times \mathbb{R}$:

   a) $G = \{[(a, b), (c, d)]: a^2 + b^2 = c^2 + d^2\}$.

   b) $H = \{[(a, b), (c, d)]: b - a = d - c\}$.

   c) $J = \{[(a, b), (c, d)]: a + b = c + d\}$.

   Find the partition corresponding to each of these equivalence relations, and describe geometrically the members of this partition. [*Hint* for (b): If $b - a = d - c = k$, note that $[(a, b), (c, d)] \in H$ if and only if (a, b) and (c, d) both satisfy the equation $y = x + k$. *Hint* for (c): If $a + b = c + d = k$, note that $[(a, b), (c, d)] \in J$ if and only if (a, b) and (c, d) both satisfy the equation $y = -x + k$.]
4. If H and J are the equivalence relations of Exercise 3, describe the equivalence relation $H \cap J$. Describe the equivalence classes modulo $H \cap J$.
5. Let H and J be the equivalence relations of Exercise 3. Prove that $H \circ J = J \circ H$; conclude that $H \circ J$ is an equivalence relation, and describe the equivalence classes modulo $H \circ J$.[*Hint*: See Exercise 10, Exercise Set 3.2.]
6. Let L be the set of all the straight lines in the plane. Let G and H be the following relations in L:

   $G = \{(\ell_1,\ell_2) : \ell_1$ is parallel to $\ell_2\}$, $H = \{\ell_1, \ell_2)$: ℓ_1 is perpendicular to $\ell_2\}$.

   Prove the following (argue informally):

a) G is an equivalence relation in L.

b) $H \circ G = H$ and $G \circ H = H$.

c) $G \cup H$ is an equivalence relation; describe its equivalence classes.

7. Let A be an arbitrary set. Prove that I_A and $A \times A$ are equivalence relations in A. Describe the partitions induced, respectively, by I_A and $A \times A$.
8. Let $\{A_i\}_{i \in I}$ be a partition of A and let $\{B_j\}_{j \in J}$ be a partition of B. Prove that $\{A_i \times B_j\}_{(i,j) \in I \times J}$ is a partition of $A \times B$.
9. Suppose $f : A \to B$ is a surjective function, and $\{B_i\}_{i \in I}$ is a partition of B. Prove that $\{\breve{f}(B_i)\}_{i \in I}$ is a partition of A.
10. Suppose $f : A \to B$ is an injective function, and $\{A_i\}_{i \in I}$ is a partition of A. Prove that $\{\bar{f}(A_i)\}_{i \in I}$ is a partition of $\bar{f}(A)$.
11. Let G and H be equivalence relations in A. Prove that each equivalence class modulo $G \cap H$ is the intersection of an equivalence class modulo G with an equivalence class modulo H. More exactly,

$$(G \cap H)_x \subseteq G_x \cap H_x, \quad \forall x \ni A.$$

12. Let G and H be equivalence relations in A, and assume that $G \cup H$ is an equivalence relation in A. Prove that each equivalence class modulo $G \cup H$ is the union of an equivalence class modulo G with an equivalence class modulo H. More exactly,

$$(G \cup H)_x = G_x \cup H_x, \quad \forall x \in A.$$

4 PRE-IMAGE, RESTRICTION AND QUOTIENT OF EQUIVALENCE RELATIONS

3.13 Definition Let $f : A \to B$ be a function, and let G be an equivalence relation in B. The *pre-image of G under f* is a relation in A defined as follows:

$$\breve{f}(G) = \{(x, y) : (f(x), f(y)) \in G\}.$$

It is simple to show that $\breve{f}(G)$ is an equivalence relation in A.

3.14 Definition Let G be an equivalence relation in A and let $B \subseteq A$. The *restriction of G to B* is a relation in B defined as follows:

$$G_{[B]} = \{(x, y) : x \in B \text{ and } y \in B \text{ and } (x, y) \in G\}.$$

It is simple to show that $G_{[B]}$ is an equivalence relation in B.

3.15 Definition Let G and H be equivalence relations in A. We call G a *refinement* of H if $G \subseteq H$; we also say that G is *finer* than H , and that H is *coarser* than G.

3.16 Theorem Let G and H be equivalence relations in A; suppose $G \subseteq H$. Then $z \in H_x \Rightarrow G_z \subseteq H_x$.

Proof. Suppose $z \in H_x$, that is $(z,x) \in H$; then we have

$$\begin{aligned} y \in G_z \Rightarrow (y, z) \in G \subseteq H \Rightarrow (y, x) \in H \qquad &\text{because we assume } (z, x) \in H \\ \Rightarrow y \in H_x. & \end{aligned}$$

Thus $G_z \subseteq H_x$. ■

3.17 Corollary If $G \subseteq H$, then for each $x \in A$, $G_x \subseteq H_x$.

This follows immediately from 3.16 and the fact that $x \in H_x$.

It follows from 3.16 that **if *G* is a refinement of *H* , then each equivalence class modulo *H* is an union of equivalence classes modulo *G***. Indeed, if H_x is an equivalence class modulo H and $z \in H_x$, then, by 3.16, H_x contains the whole class G_z; in other words, H_x contains only *whole* classes modulo G.

3.18 Definition Let G and H be equivalence relations in a set A and let G be a refinement of H. The *quotient of H by G*, which is usually denoted by H/G,is a relation in A/G defined as follows:

$$H/G = \{(G_x, G_y) : (x, y) \in H\}.$$

3.19 Theorem H/G is an equivalence relation in A/G:

Proof

H/G is reflexive: For each equivalence class G_x, $(x, x) \in H$ because H is reflexive; thus, by 3.18, $(G_x,G_x) \in H/G$.

H/G is symmetric: $(G_x,G_y) \in H/G \Rightarrow (x, y) \in H \Rightarrow (y, x) \in H \Rightarrow (G_y,G_x) \in H/G$.

H/G is transitive: $(G_x,G_y) \in H/G$ and $(G_y,G_z) \in H/G \Rightarrow (x, y) \in H$ and $(y, z) \in H \Rightarrow (x, z) \in H \Rightarrow (G_x,G_z) \in H/G$. ■

Since H/G is an equivalence relation in A/G, we may write $G_x \sim H/G\ G_y$ instead of $(G_x,G_y) \in H/G$. Thus Definition 3.18 may be written in the more suggestive form

3.20 $$G_x \underset{H/G}{\sim} G_y \quad \text{if and only if} \quad x \underset{H}{\sim} y.$$

The reader may easily verify that $G_x \underset{H/G}{\sim} G_y$ if and only if G_x and G_y are subsets of the same equivalence class modulo H.

3.21 Example Let A and G be defined as in Example 3.12; let

$$\begin{aligned} H = \{&(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (b, a), (c, d), (d, c), \\ &(c, e), (e, c), (d, e), (e, d)\}. \end{aligned}$$

It is obvious that G is a refinement of H. The partition of A induced by H is $\{H_a,H_c\}$, where $H_a =\{a, b\}$

and H_c ={c, d, e} (see Fig. 3). The reader will note that each class modulo H is a union of classes modulo G. Now A/G ={G_a,G_c,G_e};by 3.18, H/G is the following relation in A/G:

$$H/G = \{(G_a, G_a), (G_c, G_c), (G_e, G_e), (G_c, G_e), (G_e, G_c)\}.$$

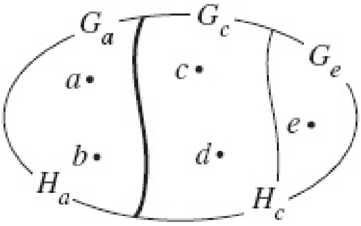


Fig.3

Note, for instance, that $G_c \underset{H/G}{\sim} G_e$. The partition of A/G induced by H/G is illustrated in Fig. 4. In particular, $(A/G)/(H/G)$ is the set {α, β}, where α ={G_a} and β = {G_c,G_e}.

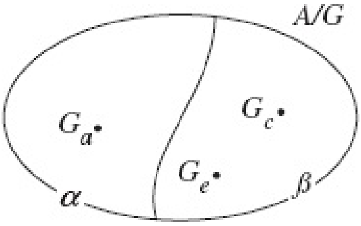


Fig.4

EXERCISES 3.4

1. Let A ={a, b, c, d, e, f }, and let G and H be the following equivalence relations in A:
   $G = I_A \cup \{(a, b), (b, a), (b, c), (c, b), (a, c), (c, a), (d, e), (e, d)\}$,
   $H = I_A \cup \{(b, c), (c, b)\}$.
   Clearly H is a refinement of G. Exhibit the sets A/G, A/H, G/H, $(A/H)/(G/H)$.
2. Let $\mathbb{R}$ be the set of the real numbers, and let G be the following relation in $\mathbb{R} \times \mathbb{R}$:

   $$G = \{[(a, b), (c, d)] : a^2 + b^2 = c^2 + d^2\}.$$

   Let $f : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ be the function given by $f(x)$ = (sin x, cos x). Describe $\check{f}$ (G); what are its equivalence classes?
3. Let $f : A \rightarrow B$ be a function and let G be an equivalence relation in B. Prove that $\check{f}$(G) is an equivalence relation in A.
4. Let $f : A \rightarrow B$ be a function and let G be an equivalence relation in B. Prove that each equivalence class modulo $\check{f}$(G) is the inverse image of an equivalence class modulo G. More precisely, if $H = \check{f}$ (G) and $y = f(x)$, prove that $H_x = \check{f}(G_y)$.
5. Let G be an equivalence relation in A and suppose that $B \subseteq A$. Prove that $G_{[B]}$ is an equivalence relation in B.
6. Let G be an equivalence relation in A and suppose $B \subseteq A$. Prove that for each $x \in B$, $(G_{[B]})_x = G_x \cap B$.

7. Let G, H , and J be equivalence relations in A, and suppose that $G \subseteq H$ and $H \subseteq J$. Prove that H/G is finer than J/G.
8. Let G, H , and J be equivalence relations in A, and suppose that $G \subseteq H$ and $H \subseteq J$. Prove each of the following.

   a) $G \subseteq H \circ J$.

   b) If $H \circ J$is an equivalence relation in A, then $(H/G) \circ (J/G) = (H \circ J)/G$.

   c) $(H/G) \circ (J/G)$ is an equivalence relation in A/G.
9. Suppose that G and H are equivalence relations in A, and that $G \subseteq H$. Prove that $G_x \underset{H/G}{\sim} G_y$ if and only if G_x and G_y are subsets of the same equivalence class modulo H.

10. Suppose that G is an equivalence relation in A, and H is an equivalence relation in B. The *product* of G and H is defined to be the following in $A \times B$:
$G \bullet H = \{[(x, w), (y, z)] : (x, y) \in G \text{ and } (w, z) \in H\}$.
Prove that $G \bullet H$ is an equivalence relation in $A \times B$.

11. Prove that every equivalence relation in a set A is the pre-image of an equivalence relation in $A \times A$. [*Hint:* Let $f : A \to A \times A$ be the function given by $f(x) = (x, x)$;if G is a relation in A, consider the relation $G \bullet G$ (see Exercise 10) in $A \times A$.]

12. Let $f : A \to B$ be a function and let G be an equivalence relation in B. Prove that $\check{f}(G) = f^{-1} \circ G \circ f$.

5 EQUIVALENCE RELATIONS AND FUNCTIONS

If $f : A \to B$ is a function, we define a relation G in A as follows:

$$G = \{(x, y) : f(x) = f(y)\}.$$

It is easy to see that G is an equivalence relation in A, G is called the *equivalence relation determined by f*.

Conversely, if G is an equivalence relation in a set A, we define a function $f : A \to A/G$ as follows:

$$f(x) = G_x, \quad \forall x \in A.$$

It is easy to see that f is a function; f is called the *canonical function from A to A/G*.

3.22 Theorem Let G be an equivalence relation in a set A. If f is the canonical function from A to A/G, then G is the equivalence relation determined by f.

Proof. Let f be the canonical function from A to A/G, and let H be the equivalence relation determined by f ; we will prove that $G = H$:

$$(x, y) \in G \Leftrightarrow G_x = G_y \Leftrightarrow f(x) = f(y) \Leftrightarrow (x, y) \in H. \ \blacksquare$$

Let A and B be sets and let $f : A \to B$ be a function; we will define three functions r, s, t, obtained from f , which play an important role in many mathematical arguments. Let G be the equivalence relation determined by f :

$r : A \to A/G$ is the canonical function from A to A/G.
$s : A/G \to \bar{f}(A)$ is the function given by $s(G_x) = f(x), \forall x \in A$.
$t : \bar{f}(A) \to B$ is the function given by $t(y) = y, \forall y \in \bar{f}(A)$.

Note that t is the inclusion function of $\bar{f}(A)$ in B (see 2.12).

3.23 Theorem Let A and B be sets, let $f : A \to B$ be a function, let G be the equivalence relation determined by f, and let r, s, t be the functions defined above. Then r is surjective, s is bijective, t is injective, and $f = t \circ s \circ r$.

Proof

i) If $G_x \in A/G$, then $x \in A$ and $r(x) = G_x$; thus r is surjective.

ii) If $f(x) \in \bar{f}(A)$, then $x \in A$, $G_x \in A/G$, and $f(x) = s(G_x)$; thus s is surjective.

iii) $s(G_x) = s(G_y) \Rightarrow f(x) = f(y) \Rightarrow (x, y) \in G \Rightarrow G_x = G_y$; thus s is injective.

iv) $t(y_1) = t(y_2) \Rightarrow y_1 = y_2$; thus t is injective.

v) Let $x \in A$; $t\{s[r(x)]\} = t[s(G_x)] = t(f(x)) = f(x)$; thus

$$[t \circ s \circ r](x) = f(x), \forall x \in A,$$

so by 2.5, $t \circ s \circ r = f$. ■

We may sum up the foregoing results by saying that any function $f : A \to B$ can be expressed as a composite of three functions r, s, t which are, respectively, surjective, bijective, and injective. This is referred to as the *canonical decomposition* of f, and it is customarily exhibited in a diagram such as the following:

$$\begin{tikzcd} A \arrow[r, "r", "\text{surj}"'] & A/G \arrow[r, "s", "\text{bij}"'] & \bar{f}(A) \arrow[r, "t", "\text{inj}"'] & B. \end{tikzcd}$$

One of the results of 3.23 is especially useful; namely, that if $f : A \to B$ is a function and G is the equivalence relation determined by f, then A/G and $\bar{f}(A)$ are in one-to-one correspondence. This is customarily expressed by writing $A/G \approx \bar{f}(A)$. In particular,

3.24 if f is surjective, then $A/G \approx B$.

Let A and B be sets, let $f : A \to B$ be a function, and let H be the equivalence relation determined by f. Let G be any equivalence relation in A which is finer than H. We define a function from A/G to B as follows:

$$[f/G](G_x) = f(x), \quad \forall x \in A. \tag{3.25}$$

It is easy to see that f/G is a function from A/G to B; f/G is called the *quotient* of f by G.

3.26 Theorem Let $f : A \to B$ be a function, let H be the equivalence relation determined by f, and let G be a refinement of H. Then H/G is the equivalence relation determined by f/G.

Proof. Let J be the equivalence relation determined by f/G; we will prove that $J = H/G$. Indeed,

$$\begin{aligned}(G_x, G_y) \in J &\Leftrightarrow [f/G](G_x) = [f/G](G_y)\\ &\Leftrightarrow f(x) = f(y)\\ &\Leftrightarrow (x, y) \in H\\ &\Leftrightarrow (G_x, G_y) \in H/G. \blacksquare\end{aligned}$$

As an example of the use of Theorem 3.26, consider the following situation: G and H are equivalence relations in A, $G \subseteq H$, f is the canonical function from A to A/H (hence, by 3.22, H is the equivalence relation determined by f). Thus, by 3.25,f/G is a function from A/G to A/H, and by 3.26, H/G is the equivalence relation determined by f/G. It is easy to see that f/G is surjective, because f is surjective. Therefore, by 3.24,

$$(A/G)/(H/G) \approx A/H.$$

EXERCISES 3.5

1. Let $f : A \to B$ be a surjective function, let G be the equivalence relation induced by f, and let H be an equivalence relation in A which is coarser than G. Define the *image of H* as follows:
   $\bar{f}(H) = \{(f(x), f(y)) : (x, y) \in H\}$
   Prove that $\bar{f}$(H) is an equivalence relation in B.
2. Let $f : A \to B$ be a surjective function, and let G be the equivalence relation induced by f. Let J be any equivalence relation in B. Prove that

   a) $\breve{f}$(J) is coarser than G.

   b) $H = \breve{f}$(J) if and only if $J = \bar{f}$(H). (See Exercise 1 above.)

   Conclude that there exists a one-to-one correspondence between the equivalence relations in B and the equivalence relations in A which are coarser than G.
3. Let $f : A \to B$ be a function, and let G be the equivalence relation determined by f. Prove that $G = f^{-1} \circ f$.
4. Let $f : A \to B$ and $g : B \to C$ be functions, and let G be the equivalence relation determined by g. Prove that $\breve{f}$(G) is the equivalence relation determined by $g \circ f$.
5. Let G and H be equivalence relations in a set A, and suppose that $G \subseteq H$. Let f be the canonical function from A to A/G. Prove that $H/G = \bar{f}$(H). (See Exercise 1.)
6. Let G and H be equivalence relations in A, and suppose that $G \subseteq H$. Let f be the canonical function from A to A/G, and let g be the canonical function from A to A/H. Let $h = g/G$. Prove that $g = h \circ f$.
7. Let G and H be equivalence relations in A, and suppose that $G \subseteq H$. If f is the canonical function from A to A/G, prove that $H = \breve{f}$(H/G).
8. Let G, H , and J be equivalence relations in A and suppose that $G \subseteq H \subseteq J$. Let $f : A \to A/G$, $g : A \to A/H$, and h: $A \to A/J$ be the canonical functions associated, respectively, with G, H , and J. Prove that $h/G = h/H \circ g/G$.
9. Let G and H be equivalence relations in A, and suppose that $G \subseteq H$. Let f be the canonical function from A to A/H. Prove that f/G is surjective.
10. Let G and H be arbitrary equivalence relations in A. Prove that

    a) $A/(G \circ H) \approx (A/G)/(G \circ H/G)$. b) $A/G \approx (A/G \cap H)/(G/G \cap H)$.

11. Let $f : A \to A$ be a function, and let G be the equivalence relation determined by f. Prove that $f \circ f = f$ if and only if

$$z \in G_x \Rightarrow f(z) \in G_x,$$

for every $z, x \in A$.

4

Partially Ordered Classes

1 FUNDAMENTAL CONCEPTS AND DEFINITIONS

By a *partially ordered class* we mean a pair of objects $\langle A, G\rangle$, where A is a class and G is an order relation in A. We say that A is ordered by G, or that G orders A. If A is a set, we say that $\langle A, G\rangle$ is a *partially ordered set*.

In ordinary mathematical applications, every partially ordered class is a partially ordered set. However, the intuitive idea of an "ordered collection of elements" is meaningful for any collection A, whether A be a set or a proper class; hence it is natural to give the definition in its most general form, letting A be any class. Once again, since very set is a class, everything we have to say about partially ordered classes applies, in particular, to partially ordered sets.

Let $\langle A, G\rangle$ be a partially ordered class; if it is well understood, in a given discussion, that G is the order relation in A, then we will say loosely that *A is a partially ordered class*. If A is a partially ordered class, ordered by G, it is customary to write $x \leqslant y$ to denote the fact that $(x, y) \in G$. We further agree that $y \geqslant x$ has the same meaning as $x \leqslant y$, and that $x \leqslant y$ means that $(x, y) \notin G$.

If $x \in A$ and $y \in A$ and $x \leqslant y$, then we say that "x is less than or equal to y." We agree that $x < y$ is an abbreviation for "$x \leqslant y$ and $x \neq y$." If $x < y$, we say that "x is strictly less than y." Note that $<$ is *not* an order relation; it is irreflexive, asymmetric, and transitive.

If A is a partially ordered class and B is a subclass of A, we may consider B to be ordered by the order relation in A. Specifically, if $x \in B$ and $y \in B$, then we let $x \leqslant y$ in B if and only if $x \leqslant y$ in A.

If A and B are partially ordered classes, there are several possible ways of ordering the class $A \times B$; the two most useful ways of doing so are given in the following definitions.

4.1 Definition Let A and B be partially ordered classes; by the *lexicographic ordering* of $A \times B$ we mean the following order relation in $A \times B$: If $(a_1,b_1) \in A \times B$ and $(a_2,b_2) \in A \times B$, then we let $(a_1,b_1) \leqslant (a_2,b_2)$ if and only if

i) $a_1 < a_2$ or

ii) $a_1 = a_2$ and $b_1 \leqslant b_2$.

The lexicographic ordering is so called because it imitates the way we order words in the dictionary (for example, *be* precedes *go* because *b* precedes *g*, and *be* precedes *by* because *e* precedes *y*).

4.2 Definition Let A and B be partially ordered classes; by the *antilexicographic ordering* of $A \times B$ we mean the following order relation in $A \times B$: If $(a_1, b_1) \in A \times B$ and $(a_2,b_2) \in A \times B$, then we let $(a_1,b_1) \leqslant (a_2,b_2)$ if and only if

i) $b_1 < b_2$ or

ii) $b_1 = b_2$ and $a_1 \leqslant a_2$.

4.3 Definition Let A be a partially ordered class. Two elements x and y in A are said to be *comparable* if either $x \leqslant y$ or $y \leqslant x$; otherwise, they are said to be *incomparable*.

4.4 Definition Let A be a partially ordered class, and let B be an arbitrary subclass of A. If every two elements of B are comparable, then we call B a *fully ordered subclass* of A,or a *linearly ordered* subclass of A, or, more commonly, a *chain* of A. If every two elements of A are comparable, then A is called a *fully ordered*,or *linearly ordered*, class.

4.5 Definition Let A be a partially ordered class and suppose $a \in A$. The *initial segment of A determined by a* is the class S_a, defined as follows:

$$S_a = \{x \in A : x < a\}.$$

4.6 Theorem Let A be a partially ordered class. If P is an initial segment of A, and Q is an initial segment of P , then Q is an initial segment of A.

Proof. By hypothesis, $P = \{x \in A : x < a\}$ for some $a \in A$, $Q = \{x \in P : x < b\}$ for some $b \in P$. Let $Q_1 = \{x \in A : x < b\}$; Q_1 is obviously an initial segment of A; we will show that $Q = Q_1$, and the theorem will thus be proved. Clearly, $Q \subseteq Q_1$; conversely, if $x \in Q_1$, then $x \in A$ and $x < b$; but $b < a$ because $b \in P$; hence $x < a$, and it follows that $x \in P$; thus $x \in Q$. ■

Theorem 4.6 may be paraphrased as follows:

An initial segment of an initial segment of *A* is an initial segment of *A*.

4.7 Definition If A is a partially ordered class, then a *cut* of A is pair (L, U) of nonempty subclasses of A with the following properties:

i) $L \cap U = \emptyset$ and $L \cup U = A$.

ii) If $x \in L$ and $y \leqslant x$, then $y \in L$.

iii) If $x \in U$ and $y \geqslant x$, then $y \in U$.

It is convenient to use a graphic device called a *line diagram* to illustrate simple properties of partially ordered classes. The elements of the class are represented by points on the diagram; if two points x and y are connected by a line, and the line rises from x to y, this means that $x \leqslant y$.

4.8 Example Fig. 1 represents a partially ordered class with six elements. Note that $\{a, b, c\}$ and $\{d, e, b, c\}$ are chains of A; $S_e = \{b, c, f\}$ is the initial segment determined by e; if $L = \{a, b, c\}$ and $U = \{d, e, f\}$, then $\{L, U\}$ is a cut.

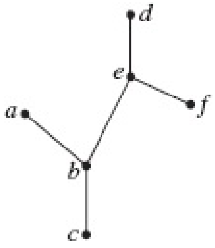
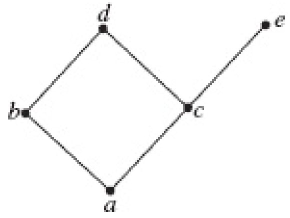


Fig. 1

4.9 Example The most important order relation in mathematics is the class inclusion relation ⊆; it is reflexive, for if A is any class, then $A \subseteq A$; it is antisymmetric, because if $A \subseteq B$ and $B \subseteq A$, then $A = B$; it is transitive, for $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. If $\mathscr{A}$ is a class (of classes) and we consider A to be ordered by the inclusion relation, we say that $\mathscr{A}$ is *ordered by inclusion*. Note that if $\mathscr{C}$ is a chain of $\mathscr{A}$, this means that for any two elements, $A, B \in \mathscr{C}$, either $A \subseteq B$ or $B \subseteq A$.

EXERCISES 4.1

1. Let $A =\{a, b, c, d\}$; if $\mathscr{P}(A)$ is ordered by inclusion, draw its line diagram.
2. Let A be the partially ordered class defined by the following diagram.



List all the chains of A, all the initial segments of A, and all the cuts of A.

3. Let A be the partially ordered class of Excercise 2. Draw the line diagram for the following classes: the class of all the chains of A (ordered by inclusion), the class of all the initial segments of A (ordered by inclusion), the class of all the cuts of A (ordered by inclusion on the “left component” L).
4. Let A and B be partially ordered classes, let C be a chain of A, and let D be a chain of B. If $A \times B$ is ordered lexicographically (4.1), prove that $C \times D$ is a chain of $A \times B$.
5. Let A and B be partially ordered classes, and let $A \times B$ be ordered antilexicographically (4.2). Prove that if (L, U) is a cut of B, then $(A \times L, A \times U)$ is a cut of $A \times B$.
6. Let A be a partially ordered class, and let G be an equivalence relation in A. Suppose the following condition holds: If $x \underset{G}{\sim} z$ and $x \leqslant y \leqslant z$, then $y \underset{G}{\sim} z$. Define a relation H in A/G by $H =\{(G_x, G_y) : \forall w \in G_x, \exists z \in G_y \ni w \leqslant z\}$. Prove that H is an order relation in A/G.

2 ORDER PRESERVING FUNCTIONS AND ISOMORPHISM

4.10 Definition Let A and B be partially ordered classes; a function $f : A \rightarrow B$ is said to be *increasing*,or *order-preserving*, if it satisfies the following condition: For every two elements $x, y \in A$,

$$x \leqslant y \Rightarrow f(x) \leqslant f(y).$$

We say that $f : A \rightarrow B$ is *strictly increasing* if it satisfies the following condition: For every two elements $x \in A$ and $y \in A$,

$$x < y \Rightarrow f(x) < f(y).$$

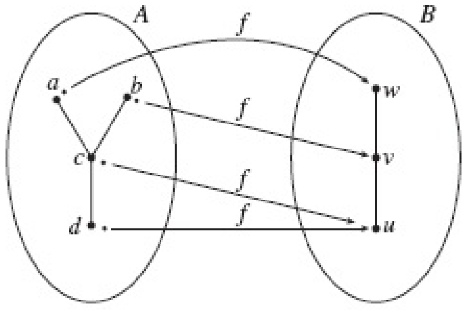
4.11 Definition Let A and B be partially ordered classes; a function $f : A \to B$ is called an *isomorphism* if it is bijective and satisfies the following condition: For every two elements $x \in A$ and $y \in A$,

$$x \leqslant y \Leftrightarrow f(x) \leqslant f(y).$$

Figures 2, 3, and 4 provide simple illustrations of the concepts we have just defined. Figure 5 describes a function which is bijective and increasing, but *is not an isomorphism* [note that $f(b)<f(a)$ but a and b are incomparable]. The reader should compare this example with Definition 4.11.

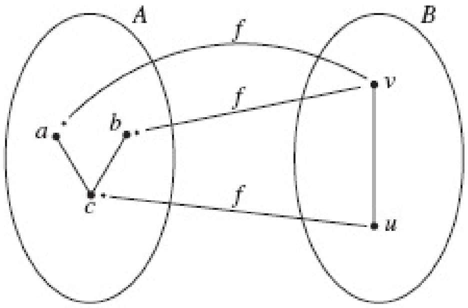
4.12 Theorem If $f : A \to B$ is an isomorphism, then

$$x < y \Leftrightarrow f(x) < f(y).$$



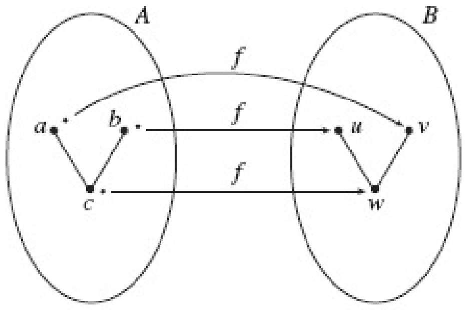
f: *A*→*B* is an increasing function

Fig. 2



f: *A*→*B* is a strictly increasing function

Fig. 3



f: *A*→*B* is an isomorphism

Fig. 4

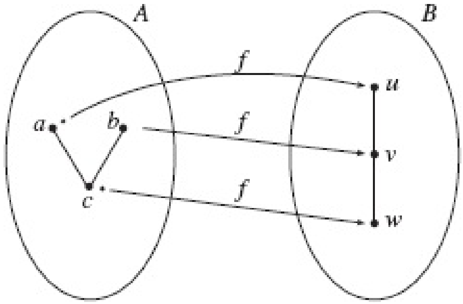


Fig. 5

Proof

i) Let us assume that $x < y$; then $x \leqslant y$; hence $f(x) \leqslant f(y)$. If $f(x) = f(y)$, then $x = y$, which is contrary to our assumption; thus $f(x) < f(y)$.

ii) The converse is proved by the same argument. ■

4.13 Theorem Let A and B be partially ordered classes and let $f : A \to B$ be a bijective function. Then $f : A \to B$ is an isomorphism if and only if $f : A \to B$ and $f^{-1} : B \to A$ are increasing functions.

Proof. Note, first, that if f is bijective, then $\forall x \in A$, $f^{-1}(f(x)) = x$. Now suppose that f and f^{-1} are increasing functions:

$$f(x) \leqslant f(y) \Rightarrow f^{-1}(f(x)) \leqslant f^{-1}(f(y)) \Rightarrow x \leqslant y;$$

from this, and the fact that f is increasing, we deduce that f is an isomorphism. Conversely, if f is an isomorphism, then certainly f is increasing; furthermore, if $f(x)$ and $f(y)$ are arbitrary elements of B, then

$$f(x) \leqslant f(y) \Rightarrow x \leqslant y \Rightarrow f^{-1}(f(x)) \leqslant f^{-1}(f(y));$$

so f^{-1} is increasing. ■

4.14 Theorem Let A, B, and C be partially ordered classes.

i) The identity function $I_A : A \to A$ is an isomorphism.

ii) If $f : A \to B$ is an isomorphism, then $f^{-1} : B \to A$ is an isomorphism.

iii) If $f : A \to B$ and $g : B \to C$ are isomorphisms, then $g \circ f : A \to C$ is an isomorphism.

Proof

i) By 2.10, $I_A : A \to A$ is bijective; now $I_A(x) = x$ and $I_A(y) = y$; hence

$$x \leqslant y \Leftrightarrow I_A(x) \leqslant I_A(y).$$

ii) If $f : A \to B$ is an isomorphism, then it is bijective, hence $f^{-1} : B \to A$ is bijective; by 4.13, $f^{-1} : B \to A$ is increasing. Finally,

$$f^{-1}(x) \leqslant f^{-1}(y) \Rightarrow f(f^{-1}(x)) \leqslant f(f^{-1}(y)) \Rightarrow x \leqslant y,$$

so $f^{-1} : B \to A$ is an isomorphism.

The proof of (iii) is left as an exercise for the reader. ■

4.15 Definition If A and B are partially ordered classes and there exists an isomorphism from A to B, we say that *A is isomorphic with B.*

Theorem 4.14 indicates that the relation "A is isomorphic with B" is an equivalence relation among partially ordered classes. Indeed, by 4.14(i), A is isomorphic with A; by 4.14(ii), if A is isomorphic with B, then B is isomorphic with A; by 4.14(iii), if A is isomorphic with B and B is isomorphic with C, then A is isomorphic with C.

We will write $A \cong B$ to denote the fact that A is isomorphic with B.

The concept of isomorphism is of great importance in the study of partially ordered classes. Suppose that A and B are partially ordered classes and $f : A \to B$ is an isomorphism; let us agree to write x' instead of $f(x)$; since $f : A \to B$ is bijective, every element x in A corresponds with a unique element x' in B:

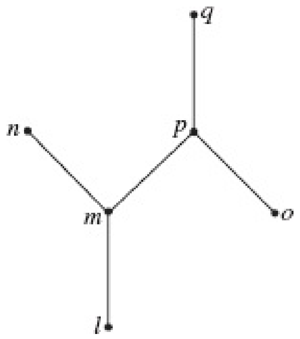
$$\begin{aligned} &x \overset{f}{\longmapsto} x', \\ &y \overset{f}{\longmapsto} y', \\ &z \overset{f}{\longmapsto} z', \quad \text{etc} \ldots. \end{aligned}$$

Furthermore, by 4.12, if x and y are any two elements in A, then

$$x < y \Leftrightarrow x' < y'.$$

This means that the ordering of A is exactly the same as the ordering of B; in particular (if it is practical to draw the line diagrams of A and B), the line diagram of A is the same as the line diagram of B. Thus, essentially, there is no difference between A and B except the letters we use to designate their elements.

4.16 Example If $B = \{l, m, n, o, p, q\}$ is ordered as in the following diagram,



then B is isomorphic with the class A of Example 4.8. The isomorphism $f : A \to B$ is given by the following table.

4.17

x	$f(x)$
a	n
b	m
c	l
d	q
e	p
f	o

In conclusion, if A is isomorphic with B, and if we identify corresponding elements, then A and B are essentially the same partially ordered class.

EXERCISES 4.2

1. Prove that if $f : A \to B$ is an injective, increasing function, then it is strictly increasing.
2. Let $f : A \to B$ be an increasing function. If C is a chain of A, prove that $\bar{f}(C)$ is a chain of B.
3. Let A and B be partially ordered classes. Prove that if $A \times B$ is ordered lexicographically, then the projection function $p_1 : A \times B \to A$ is increasing; if $A \times B$ is ordered antilexicographically, then the projection function $p_2 : A \times B \to B$ is increasing. [Note that $p_1(x, y) = x$, $p_2(x, y) = y$.]
4. A subclass C of a partially ordered class is called *convex* if it satisfies the following condition: If $a \in C$ and $b \in C$ and $a \leqslant x \leqslant b$, then $x \in C$. Let A and B be partially ordered classes, let $f : A \to B$ be an increasing function, and let C be a convex subclass of B. Prove that $\breve{f}(C)$ is a convex subclass of A.
5. Let A and B be partially ordered classes, let $f : A \to B$ be an increasing function, and let H be the equivalence relation determined by f. Prove that each equivalence class modulo H is a convex subclass of A.
6. Let A and B be partially ordered classes, and let $f : A \to B$ be an increasing function; assume $\bar{f}(A) = B$. Prove that if (L, U) is a cut of B, then $(\breve{f}(L), \breve{f}(U))$ is a cut of A.
7. Prove that the composite of two increasing functions is increasing. Use this result to prove 4.14(iii).
8. Let A and B be partially ordered classes, and let $f : A \to B$ be an isomorphism. Prove each of the following:
    a) If C is a convex subclass of A, then $\bar{f}(C)$ is a convex subclass of B.
    b) If (L, U) is a cut of A, then $(\bar{f}L), \bar{f}(U))$ is a cut of B.
    c) If $[a, b]$ is a closed interval of A, then $\bar{f}([a, b])$ is a closed interval of B. (If $a, b \in A$, then the set $\{x \in A : a \leqslant x \leqslant b\}$ is called a *closed interval* of A and is denoted by the symbol $[a, b]$.)
9. Let E and F be partially ordered classes, and let $g : E \to F$ be an isomorphism. Prove that for arbitrary $x \in E$, $\bar{g}(S_x) = S_{g(x)}$; conclude that $S_x \approxeq S_{g(x)}$.
10. Let A be a partially ordered set. For each $a \in A$, let $I_a = \{x \in A : x \leqslant a\}$. Let $\mathscr{I} = \{I_a\}_{a \in A}$ and let $\mathscr{I}$ be ordered by inclusion. Prove that $\mathscr{I}$ is isomorphic with A.
11. Let A, B, and C be mutually disjoint, partially ordered classes. If $A \cong B$, prove that $(A \cup C) \cong (B \cup C)$.[*Hint*: Take the union of two functions, as in 2.16.]
12. Let A be a partially ordered set. Define $L_x = \{z \in A : z \leqslant x\}$ and $U_x = \{z \in A : z \nleqslant x\}$. Prove the following:

a) For each $x \in A$, (L_x, U_x) is a cut of A.

b) The function ϕ defined by $\phi(x) = (L_x, U_x)$ is an isomorphism between A and the class of all the cuts of the above-described form. The set of cuts (L, U) is ordered by inclusion on L.

3 DISTINGUISHED ELEMENTS. DUALITY

Certain distinguished elements play an important part in the study of partially ordered classes. We now define them; in each of the following definitions, we assume that A is a partially ordered class.

4.18 Definition An element $m \in A$ is called a *maximal element* of A if none of the elements of A are strictly greater than m; in symbols, this can be expressed as follows:

$$\forall x \in A, \quad \text{if } x \geqslant m, \quad \text{then } x = m.$$

Similarly, an element $n \in A$ is called a *minimal element* of A if none of the elements of A are strictly less than n; in symbols,

$$\forall x \in A, \quad \text{if } x \leqslant n, \quad \text{then } x = n.$$

4.19 Definition An element $a \in A$ is called the *greatest element* of A if $a \geqslant x$ for every $x \in A$. An element $b \in A$ is called the *least element* of A if $b \leqslant x$ for every $x \in A$.

It is easy to see that if A has a greatest element, then this element is unique; for suppose a and a' are both greatest elements of A. Then $a \leqslant a'$ and $a' \leqslant a$; hence $a = a'$. Analogously, the least element of A is unique.

4.20 Definition Let B be a subsets of A. An *upper bound of B in A* is an element $a \in A$ such that $a \geqslant x$ for every $x \in B$.A *lower bound of B in A* is an element $b \in A$ such that $b \leqslant x$ for every $x \in B$. When there is no risk of ambiguity, we will refer to an upper bound of B in A simply as an “*upper bound of B*,” and to a lower bound of B in A simply as a “*lower bound of B*.” The class of all the upper bounds of B will be denoted by $\upsilon(B)$ and the class of all the lower bounds of B will be denoted by $\lambda(B)$.

4.21 Definition If the class of lower bounds of B in A has a greatest element, then this element is called the *greatest lower bound* of B in A. If the class of upper bounds of B in A has a least element, then this element is called the *least upper bound* of B in A. The least upper bound of B in A is also called the *supremum* of B in A (abbreviated $\sup_A$ B), and the greatest lower bound of B in A is also called the *infinum* of B in A (abbreviated $\inf_A$ B). When there is no risk of ambiguity, we will write sup B for $\sup_A$ B, and inf B for $\inf_A$ B.

We have seen that the greatest element and the least element of any class are unique; hence the sup and the inf, if they exist, are unique.

Examples

4.22 Figure 6 is the line diagram of a class that has maximal elements but no greatest element (a and d are maximal elements).

4.23 In Fig. 7, let $A = \{a, b, c, d, e, f\}$ and let $B = \{b, c, e, f\}$. B has two upper bounds in A, namely a and d, but no sup.

4.24 In Fig. 7, let A and B be defined as above, and let $C = \{a, b, c, e, f\}$ and $D = \{d, b, c, e, f\}$. Then B has no sup in A, although $\sup_C B = a$ and $\sup_D B = d$.

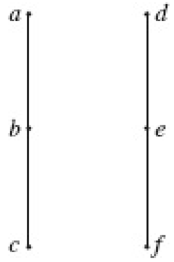


Fig. 6

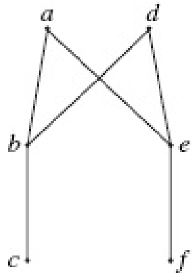


Fig. 7

4.25 The class N of all the positive integers has a least element but no greatest element and no maximal elements. The class $\mathbb{Z}$ of all the integers has neither a greatest nor a least element.

4.26 Let $\mathscr{A}$ be a class (of classes) which is ordered by inclusion; let $\mathscr{B} = \{B_i\}_{i \in I}$ be a subclass of $\mathscr{A}$, and let us assume that $\bigcup_{i \in I} B_i$ and $\bigcap_{i \in I} B_i$ are elements of $\mathscr{A}$. Then $\sup \mathscr{B} = \bigcup_{i \in I} B_i$: indead,each B_i is $\subseteq \bigcup_{i \in I} B_i$, hence $\bigcup_{i \in I} B_i$ is an upper bound of $\mathscr{B}$; furthermore, if C is any other upper bound of $\mathscr{B}$, this means that $B_i \subseteq C$ for every $i \in I$; hence, by 1.40(i), $\bigcup_{i \in I} B_i \subseteq C$; this proves that $\bigcup_{i \in I} B_i$ is the least upper bound of $\mathscr{B}$. Similarly, inf $\mathscr{B} = \bigcap_{i \in I} B_i$, for clearly $\bigcap_{i \in I} B_i$ is $\subseteq$ each B_i, hence $\bigcap_{i \in I} B_i$ is a lower bound of $\mathscr{B}$; furthermore, if D is any other lower bound of $\mathscr{B}$, this means that $D \subseteq B_i$ for every $i \in I$, hence, by 1.40(ii), $D \subseteq \bigcap_{i \in I} B_i$; this proves that $\bigcap_{i \in I} B_i$ is the greatest lower bound of $\mathscr{B}$.

4.27 It is important to note that $\lambda(\varnothing) = A$. Indeed, if $x \in A$, then the statement "$x \leqslant y$ for every $y \in \varnothing$" is not false (for to deny it would be to assert that "$\exists y \in \varnothing\ x \nleqslant y$," which is absurd); hence it is true. This holds for each element $x \in A$, so we conclude that $\lambda(\varnothing) = A$. Next, we note that inf $\varnothing$ is the greatest element of $\lambda(\varnothing)$; thus if A has a greatest element a, then $a = \inf \varnothing$. Analogously, if b is the least element of A, then $b = \sup \varnothing$.

When we reason about partially ordered classes, we are led to the interesting notion of *duality*. Briefly, duality can be explained as follows.

If G is an order relation in A, then $G^{-1}$is also an order relation in A (see Exercise 8, Exercise Set 3.2). Let $\langle A, G\rangle$ refer to the class A ordered by G, and let $\langle A, G^{-1}\rangle$ refer to A ordered by G^{-1}. Then $x \leqslant y$ in $\langle A, G\rangle$ if and only if $x \geqslant y$ in $\langle A, G^{-1}\rangle$; it follows that a is a maximal element of $\langle A, G\rangle$ if and only if a is a minimal element of $\langle A, G^{-1}\rangle$; a is the greatest element of $\langle A, G\rangle$ if and only if a is the least element of $\langle A, G^{-1}\rangle$; if $B \subseteq A$, then b is an upper bound of B in $\langle A, G\rangle$ if and only if b is a lower bound of B in $\langle A, G^{-1}\rangle$; and $b = \sup B$ in $\langle A, G\rangle$ if and only if $b = \inf B$ in $\langle A, G^{-1}\rangle$.

Let $\mathscr{S}$ be a statement about partially ordered classes. In $\mathscr{S}$, suppose that we replace each occurrence of $\leqslant$ by $\geqslant$ and vice versa; suppose furthermore that we replace the words “maximal” by “minimal” and vice versa, “greatest” by “least” and vice versa, “upper bound” by “lower bound” and vice versa, “sup” by “inf” and vice versa. The resulting statement $\mathscr{S}'$ 'is called the *dual* of $\mathscr{C}$.

In view of what we have said above, it is easy to see that if $\mathscr{S}$ is a true statement in A, G^{-1} , then the dual of $\mathscr{S}$ is true in A, G. In particular, suppose that $\mathscr{S}$ is a theorem for all partially ordered classes. If A is any partially ordered class and G is the order relation in A, then $\mathscr{S}$ is true in A, G^{-1} , hence the dual of $\mathscr{S}$ is true in A, G. Thus if$\mathscr{S}$ is a theorem for all partially ordered classes, the dual of$\mathscr{S}$ is also a theorem.

The concept of duality permits a considerable economy in the presentation of theorems about partially ordered classes, for every time we prove a theorem, we know that the dual of the theorem is also true.

The following are a few properties of the distinguished elements in a partially ordered class.

4.28 Theorem If A has a greatest element a, and B has a greatest element b, and $A \subseteq B$, then $a \leqslant b$.

Proof. By definition, $b \geqslant x$ for every $x \in B$;but $a \in A \subseteq B$; hence $b \geqslant a$. ■

Dual If A has a least element a, and B has a least element b, and $A \subseteq B$, then $a \geqslant b$.

4.29 Theorem Let B and C be subclasses of A. If $B \subseteq C$, then $v(C) \subseteq v(B)$.

Proof. $x \in v(C) \Rightarrow x \geqslant y, \forall y \in C \Rightarrow x \geqslant y, \forall y \in B \Rightarrow x \in v(B)$. ■

Dual If $B \subseteq C$, then $\lambda(C) \subseteq \lambda(B)$.

4.30 Theorem Let B and C be subclasses of A, and suppose that B and C each has a sup in A. If $B \subseteq C$ the sup $B \leqslant$ sup C.

Proof. By 4.29, $v(C) \subseteq v(B)$; hence by 4.28 (dual), sup $B \leqslant$ sup C. ■

Dual If B and C each has a inf in A, and if $B \subseteq C$, then inf $B \geqslant$ inf C.

4.31 Theorem Let B be a subclass of A. Then $B \subseteq v(\lambda(B))$.

Proof. Suppose $x \in B$; for each $y \in \lambda(B)$, $y \leqslant x$, that is, $x \geqslant y$; hence $x \in v(\lambda(B))$. ■

Dual $B \subseteq \lambda(v(B))$.

4.32 Lemma Let B be a subclass of A and suppose that $\lambda(B)$ has a sup in A. Then B has an inf in A, and inf B = sup $\lambda(B)$.

Proof. Let a = sup $\lambda(B)$. Suppose $b \in B$; for every $c \in \lambda(B)$, $c \leqslant b$; hence b is an upper bound of $\lambda(B)$; thus $a \leqslant b$. This is true for each $b \in B$, so we conclude that *a is a lower bound of B*. Now if d is *any* lower bound of B, then $d \in \lambda(B)$,so $a \geqslant d$ because a is an upper bound of $\lambda(B)$. We have proved that a is the greatest lower bound of B. ■

Dual If $v(B)$ has an inf in A, then B has a sup in A and sup B = inf $v(B)$.

4.33 Definition Let A be a partially ordered class. If every nonempty subclass of A that is bounded above has a sup, then A is said to be *conditionally complete*.

We have the following alternative definition of *conditionally complete*: A is called conditionally complete if every nonempty subclass of A that is bounded below has an inf. Our next theorem establishes the equivalence of the two definitions.

4.34 Theorem The following two conditions are equivalent:

i) Every nonempty subclass of A that is bounded above has a sup.

ii) Every nonempty subclass of A that is bounded below has an inf.

Proof

a) Suppose that (i) holds; let B be a nonempty subclass of A which is bounded below, that is, $\lambda(B) \neq \varnothing$. Each element of B is an upper bound of $\lambda(B)$, hence $\lambda(B)$ is bounded above; thus $\lambda(B)$ has a sup. But, by 4.32, it follows that B has an inf.

b) The converse is the dual of the result we have just proven. ■

EXERCISES 4.3

1. Suppose $B \subseteq A$; prove that if B has a greatest element b, then b = sup B.
2. Suppose $B \subseteq A$; prove that $v(B) = v(\lambda(v(B)))$.[*Hint*: Use 4.29 and 4.31.]
3. Suppose $B \subseteq A$ and $C \subseteq A$; prove that $\lambda(B \cup C) = \lambda(B) \cap \lambda(C)$.
4. Suppose $B \subseteq A$; prove that if B has a sup b, then $\lambda(v(B)) \cap v(B) =\{b\}$.
5. Suppose $C \subseteq B$ and $B \subseteq A$; prove that $\sup_A C \leqslant \sup_B C$.
6. Let B and C be subclasses of a partially ordered class A. Prove that if sup B = sup C, then $v(B) = v(C)$.
7. Let A and B be partially ordered classes and let $f : A \to B$ be a strictly increasing function. Prove that if b is a maximal element of B, then each element of $\check{f}(b)$ is a maximal element of A.
8. Let A and B be partially ordered classes, and let $f : A \to B$ be an increasing function. Prove that if a is the greatest element of A, then $f(a)$ is the greatest element of $\bar{f}(A)$.
9. Let A and B be partially ordered classes, and let $f : A \to B$ be an increasing function; suppose $C \subseteq A$. Prove that if c is an upper bound of C, then $f(c)$ is an upper bound of $\bar{f}(C)$.
10. Let A and B be partially ordered classes, and let $f : A \to B$ be an isomorphism. Prove each of the

following:

a) a is a maximal element of A iff $f(a)$ is a maximal element of B.

b) a is the greatest element of A iff $f(a)$ is the greatest element of B.

c) Suppose $C \subseteq A$; x is an upper bound of C iff $f(x)$ is an upper bound of $\bar{f}(C)$.

d) $b = \sup C$ iff $f(b) = \sup \bar{f}(C)$.

11. Let A be a partially ordered class. Prove the following:

a) If every subclass of A has a sup and an inf, in A, then A has a least element and a greatest element. [*Hint*: Use 4.27.]

b) The following two statements are equivalent: Every subclass of A has a sup; every subclass of A has an inf.

12. Let A and B be partially ordered classes. Prove the following:

a) Suppose $A \times B$ is ordered lexicographically: if (a, b) is a maximal element of $A \times B$, then a is a maximal element of A.

b) Suppose $A \times B$ is ordered antilexicographically. If (a, b) is a maximal element of $A \times B$, then b is a maximal element of B.

4 LATTICES

4.35 Definition Let A be a partially ordered class. If every doubleton $\{x, y\}$ in A has a sup and an inf, then A is called a *lattice*.

When dealing with lattices it is customary to denote $\sup\{x, y\}$ by $x \vee y$ and $\inf\{x, y\}$ by $x \wedge y$. If A is a lattice, $x \vee y$ is often called the *join* of x and y, and $x \wedge y$ is often called the *meet* of x and y; the expression $x \vee y$ is read "x join y" and the expression $x \wedge y$ is read "x meet y."

Note the following simple consequences of our definition. If a and b are arbitrary elements of a lattice A, then

4.36 $a \leqslant a \vee b$ and $b \leqslant a \vee b$

because $a \vee b$ is an upper bound of a and b. Furthermore, if $c \in A$, then

4.37 $a \leqslant c$ and $b \leqslant c \Rightarrow a \vee b \leqslant c$;

in other words, if c is an upper bound of a and b, then $a \vee b \leqslant c$ because $a \vee b$ is the *least* upper bound of a and b.

For analogous reasons, we have

4.38 $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$

4.39 $c \leqslant a$ and $c \leqslant b \Rightarrow c \leqslant a \wedge b$;

4.40 Theorem Let A be a lattice; the join and the meet have the following properties:

L1. $x \vee x = x$ and $x \wedge x = x$.

L2. $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$.

L3. $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.

L4. $(x \vee y) \wedge x = x$ and $(x \wedge y) \vee x = x$.

Proof. L1 and L2 are immediate consequences of the definitions of sup and inf.

L3. First, we will prove that

$$(x \vee y) \vee z \leqslant x \vee (y \vee z);$$

by 4.36,

$$x \leqslant x \vee (y \vee z) \quad \text{and} \quad y \leqslant y \vee z \leqslant x \vee (y \vee z),$$

hence, by 4.37, $x \vee y \leqslant x \vee (y \vee z)$ furthermore, by 4.36,

$$z \leqslant y \vee z \leqslant x \vee (y \vee z),$$

so by 4.37, $(x \vee y) \vee z \leqslant x \vee (y \vee z)$. The inequality $x \vee (y \vee z) \leqslant (x \vee y) \vee z$ is proven in the same way, hence

$$(x \vee y) \vee z = x \vee (y \vee z).$$

The dual of this result is $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.

L4. To prove that $(x \vee y) \wedge x = x$ is to prove that

$$x = \inf\{x \vee y, x\}.$$

Now x is a lower bound of $\{x \vee y, x\}$ because $x \leqslant x \vee y$ by 4.36 and obviously $x \leqslant x$. Furthermore, if z is any lower bound of $\{x \vee y, x\}$, then $z \leqslant x$; thus x is the greatest lower bound of $\{x \vee y, x\}$. This proves that $(x \vee y) \wedge x = x$; the dual of this result is $(x \wedge y) \vee x = x$. ■

A lattice may alternatively be defined as an algebraic system with two operations $\vee$ and $\wedge$ which have properties L1 through L4. This fact, which is of great importance in the study of lattices, is a consequence of the following theorem.

4.41 Theorem Let A be a class in which two operations denoted $\vee$ and $\wedge$ are given and have properties L1 through L4. We define a relation in A, to be denoted by the symbol $\leqslant$, as follows:

4.42 $x \leqslant y$ if and only if $x \vee y = y$.

Then $\leqslant$ is an order relation in A, and A is a lattice.

Proof. First, let us prove that the relation $\leqslant$ defined above is an order relation.

Reflexive. By L1, $x \vee x = x$; hence, by 4.42, $x \leqslant x$.

Antisymmetric. Suppose that $x \leqslant y$ and $y \leqslant x$; by 4.42, $x \vee y = y$ and $y \vee x = x$; but by L2, $x \vee y = y \vee x$, hence, $x = y$.

Transitive. Suppose that $x \leqslant y$ and $y \leqslant z$; by 4.42, $x \vee y = y$ and $y \vee z = z$; thus

$$x \vee z = x \vee (y \vee z) = (x \vee y) \vee z = y \vee z = z;$$

so by 4.42, $x \leqslant z$.

Next, we will prove that $x \vee y$ is the least upper bound of x and y:

$$x \vee (x \vee y) = (x \vee x) \vee y = x \vee y,$$

so by 4.42, $x \leqslant x \vee y$; analogously, $y \leqslant x \vee y$, hence $x \vee y$ is an upper bound of x and y. Now if z is any upper bound of x and y, that is, $x \leqslant z$ and $y \leqslant z$, then $x \vee z = z$ and $y \vee z = z$; thus

$$(x \vee y) \vee z = x \vee (y \vee z) = x \vee z = z,$$

so by 4.42, $x \vee y \leqslant z$. This proves that $x \vee y = \sup\{x, y\}$. The proof that $x \wedge y = \inf\{x, y\}$ is left as an exercise for the reader. We conclude that A is a lattice. ■

It follows from 4.40 and 4.41 that a lattice may be defined in two distinct ways: as a partially ordered class in which every pair of elements has a sup and an inf or, alternatively, as an algebraic system with two operations satisfying rules L1 through L4.

4.43 Definition Let A be a lattice, and let B be a subclass of A. If

$$x \in B \text{ and } y \in B \Rightarrow x \vee y \in B \text{ and } x \wedge y \in B,$$

then B is called a *sublattice* of A.

4.44 Definition A *Boolean algebra* is defined to be a lattice A with the following additional properties:

L5. There is an element $0 \in A$ and an element $1 \in A$ such that for each $x \in A$, $x \vee 0 = x$ and $x \wedge 1 = x$.

L6. For each $x \in A$ there is an element $x' \in A$ such that

$$x \wedge x' = 0 \quad \text{and} \quad x \vee x' = 1.$$

L7. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

The algebra of classes is an example of a Boolean algebra; indeed, by 1.22, 1.25, and 1.26, the operations $\cup$, $\cap$ and $'$ satisfy L1 through L7. We have noted independently (4.26) that $A \cup B = \sup\{A, B\}$ and $A \cap B = \inf\{A, B\}$. Another example of a Boolean algebra is the algebra of sentences, with the operations of conjunction, disjunction, and negation. In addition, Boolean algebra has a variety of applications in many areas of science and technology.

4.45 Definition Let A be a partially ordered class; A is called a *complete lattice* if every subclass of A has a sup. Alternatively, A is called a complete lattice if every subclass of A has an inf.

The purpose of our next theorem is to show that these two alternative definitions are equivalent. It will follow that if A is a complete lattice (in the sense of either definition), then every subclass of A has a sup and an inf; in particular, every doubleton in A has a sup and an inf, hence we are justified in calling A a lattice.

4.46 Theorem Let A be a partially ordered class; the following two conditions are equivalent:

i) every subclass of A has a sup;

ii) every subclass of A has an inf.

Proof. Let us assume that (i) holds; it follows that A has a sup, which is necessarily the greatest element of A, and $\varnothing$ has a sup, which is the least element of A (see 4.27). Let M designate the greatest element of A, and let m designate the least element of A. Let B be an arbitrary subclass of A. If $B = \varnothing$, then inf $B = M$ (see 4.27);

if $B \neq \varnothing$, then B is bounded below by m, hence by Theorem 4.34, B has an inf. Thus, (ii) holds; the converse is the dual of what we have just proven. ■

4.47 Example Let A be an arbitrary set and let $\mathscr{G}$ be the set of all the equivalence relations in A, ordered by inclusion. $\mathscr{G}$ has a least element, namely the relation I_A, and a greatest element, namely the relation $A \times A$. Furthermore, if $\{G_i\}_{i\in I}$ is any subset of $\mathscr{G}$, then $\bigcap_{i\in I} G_i$ is an element of $\mathscr{G}$ (see Exercise 4, Exercise Set 3.2); as we have seen (4.26), $\bigcap_{i\in I} G_i$ is the greatest lower bound of $\{G_i\}_{i\in I}$. Thus, $\mathscr{G}$ is a complete lattice by 4.46(ii).

EXERCISES 4.4

1. Let A be a class with two operations $\vee$ and $\wedge$ which satisfy L1 through L4. Prove that $x \vee y = y$ if and only if $x \wedge y = x$.
2. In Theorem 4.41, prove that $x \wedge y = \inf\{x, y\}$. [*Hint*: Use the result of Exercise 1.]
3. Let A be a lattice. Prove that the following statements are true.
   a) If $a \leqslant b$, then $\forall x \in A$, $a \vee x \leqslant b \vee x$ and $a \wedge x \leqslant b \wedge x$.
   b) If $a \leqslant b$ and $c \leqslant d$, then $a \vee c \leqslant b \vee d$ and $a \wedge c \leqslant b \wedge d$.
4. Let A be a lattice. If $[a, b]$ and $[c, d]$ are closed intervals of A, prove that

$$[a, b] \cap [c, d] = [a \vee c, b \wedge d].$$

   (See Exercise 8, Exercise Set 4.2, for a definition of closed interval.)
5. Let A be a lattice; prove that every closed interval $[a, b]$ of A is a sublattice of A.
6. Let A be a lattice; if $a \in A$, let $I_a = \{x \in A : x \leqslant a\}$. Prove that I_a is a sublattice of A.
7. By a distributive lattice we mean a class with two operations satisfying L1 through L4 and L7. If A is a distribute lattice, prove that

$$c \vee x = c \vee y \text{ and } c \wedge x = c \wedge y \Rightarrow x = y.$$

8. In an arbitrary lattice *A*, prove the so-called “distributive inequalities”

$$x \wedge (y \vee z) \geqslant (x \wedge y) \vee (x \wedge z) \quad \text{and} \quad x \vee (y \wedge z) \leqslant (x \vee y) \wedge (x \vee z).$$

9. Let *A* be a lattice and let $x, y, z \in A$. Prove that if $x \leqslant z$, then

$$x \vee (y \wedge z) \leqslant (x \vee y) \wedge z.$$

10. Draw the line diagram of a lattice *A* and a subclass $B \subseteq A$ such that *B* is not a sublattice of *A*.
11. Draw the line diagram of two lattices whose intersection is not a lattice.
12. Draw the line diagram of a lattice which is not a Boolean algebra.
13. Let *A* be a partially ordered set; prove that the class of all the cuts of *A* (ordered by inclusion on the “left components” *L*) is a complete lattice.
14. Let *A* be a partially ordered set; prove that the class of all the convex subsets of *A* is a complete lattice.
15. Draw the line diagram of a partially ordered class which is conditionally complete, but is not a complete lattice.

5 FULLY ORDERED CLASSES. WELL-ORDERED CLASSES

Let *A* be a partially ordered class. As previously stated, *A* is said to be *fully ordered* if every two elements of *A* are comparable.

Examples of fully ordered classes are: the class $\mathbb{Z}$ of the integers, the class $\mathbb{Q}$ of the rational numbers, and the class $\mathbb{R}$ of the real numbers. One can easily see that every subclass of a fully ordered class is fully ordered. It is evident, too, that every fully ordered class is a lattice.

4.48 Theorem Let $f : A \to B$ be a function, where *A* is a fully ordered class and *B* is a partially ordered class. If $f : A \to B$ is bijective and increasing, it is an isomorphism.

Proof. Suppose $f(x) \leqslant f(y)$; since *x* and *y* are comparable, either $x \leqslant y$ or $y < x$. If $y < x$, then $f(y) \leqslant f(x)$; but $f(y) = f(x)$ would imply $y = x$, so we must have $f(y) < f(x)$. This is contrary to our assumption, hence $x \leqslant y$. ■

4.49 Definition Let *A* be a partially ordered class. *A* is said to be *well ordered* if every nonempty subclass of *A* has a least element.

If *A* is well ordered, then *A* is fully ordered; for if $x \in A$ and $y \in A$, then the doubleton $\{x, y\}$ has a least element, which is either *x* or *y*; hence $x \leqslant y$ or $y \leqslant x$.

If *A* is well ordered, then *A* is conditionally complete; for if *B* is a subclass of *A* and $v(B) \neq Ø$, then $\upsilon(B)$ has a least element which is by definition the sup of *B*.

4.50 Definition Let *A* be a partially ordered class, and suppose $a \in A$. An element $b \in A$ is called the *immediate successor* of *a* if $a < b$ and there is no element *c* in *A* such that $a < c < b$.

4.51 *Remark.* If A is a well-ordered class, then every element of A (with the exception of the greatest element of A, if it exists) has an immediate successor. Indeed, if $x \in A$ and x is not the greatest element of A, then the class $T = \{y \in A : y > x\}$ is nonempty, hence T has a least element which is obviously the immediate successor of x.

Let A be a nonempty well-ordered class. By 4.49, A has a least element, which may be denoted by x_1; if x_1 is not the greatest element (that is, the only element) of A, then by 4.51, x_1 has an immediate successor, which may be denoted by x_2; again, if x_2 is not the greatest element of A, then by 4.51, x_2 has an immediate successor, which may be denoted by x_3; and so on.

Examples

4.52 The class of numbers {1, 2, 3, 4, 5}, ordered in the usual way, is a well-ordered class.

4.53 The class N ={1, 2, 3, 4, …} of all the positive integers, ordered in the usual way, is a well-ordered class.

4.54 $W = \{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots, 1, 1+\frac{1}{2}, 1+\frac{3}{4}, 1+\frac{7}{8}, \ldots, 2, 2+\frac{1}{2}, 2+\frac{3}{4}, 2+\frac{7}{8}, \ldots\}$, ordered in the usual way, is a well-ordered subclass of the real numbers.

We may define y to be an *immediate predecessor* of x if and only if x is an immediate successor of y; note that in Examples 4.52 and 4.53, only the least element of the class (namely 1) does not have an immediate predecessor; however, in Example 4.54, there are three elements (namely 0, 1 and 2) which do not have an immediate predecessor.

Note that the class $\mathbb{Z}$ of all the integers, the class $\mathbb{Q}$ of the rational numbers, and the class $\mathbb{R}$ of the real numbers are *not* well ordered. Indeed,

$$V = \{\ldots, -3, -2, -1, 0\}$$

is a subclass of each of these classes, and V does not have a least element.

4.55 Definition Let A be a partially ordered class; we define a *section* of A to be a subclass $B \subseteq A$ with the following property:

$$\forall x \in A, \quad \text{if } y \in B \text{ and } x \leqslant y \quad \text{then} \quad x \in B.$$

4.56 Theorem Let A be a well-ordered class; B is a section of A if and only if $B = A$ or B is an initial segment of A.

Proof

i) If $B = A$ or B is an initial segment of A, then obviously B is a section of A.

ii) Conversely, suppose B is a section of A; if $B = A$, we are done; thus, suppose $B \neq A$, that is, $A - B \neq$ Ø. Because A is a well-ordered class, $A - B$ has a least element which we denote by m; we will show that $B = S_m$. Well,

$$x \in S_m \Rightarrow x < m \Rightarrow x \in B$$

(because m is the *least* element of $A - B$); conversely, suppose $x \in B$: if $m \leqslant x$, then $m \in B$ by 4.55, and this contradicts our choice of m; thus $x < m$, so $x \in S_m$. ■

One of the most important features of well-ordering is the fact that induction can be used to prove theorems about all the elements of a well-ordered class. This fact is given in the following theorem.

4.57 Theorem (*Principle of Transfinite Induction*). Let A be a well-ordered class, and let $P(x)$ be a statement which is either true or false for each element $x \in A$; suppose the following condition holds:

Ind. If $P(y)$ is true for every $y < x$, then $P(x)$ is true.

In that case, $P(x)$ is true for every element $x \in A$.

Proof. Suppose that $P(x)$ is *not* true for every $x \in A$; then the class $\{y \in A : P(y) \text{ is false}\}$ is nonempty; hence, by 4.49, has a least element m. Now $P(x)$ is true for every $x < m$, so by Ind, $P(m)$ is true; but we chose m to be the least element of $\{y \in A : P(y) \text{ is false}\}$, so $P(m)$ is false. This contradiction proves that $P(x)$ must be true for every $x \in A$. ■

EXERCISES 4.5

1. Let A be a fully ordered set. Prove that the set of all sections of A (ordered by inclusion) is fully ordered.
2. Let A be a fully ordered class, let B be a partially ordered class, and let $f : A \to B$ be an increasing function. Prove that f is injective if and only if f is strictly increasing.
3. Let A be fully ordered class, let B be a partially ordered class, and let $f : A \to B$ be a bijective function. Prove that if f is increasing, then f is an isomorphism.
4. Let A be a fully ordered class and let $\{L, U\}$ be a partition of A. Prove that (L, U) is a cut of A if and only if $\forall x \in L$ and $\forall y \in U$, $x \leqslant y$.
5. Let A be a fully ordered class. Prove that if B and C are convex subclasses of A and $B \cap C \neq \varnothing$, then $B \cup C$ is convex.
6. Let A be a fully ordered class. Let B and C be convex subclasses of A and suppose $B \cap C \neq \varnothing$. Prove that every upper bound of $B \cap C$ is an upper bound of B or an upper bound of C. Conclude that

   a) $a \leqslant b$ if and only if $a' \leqslant b'$. b) $a = b$ if and only if $a' = b'$.
   c) $a < b$ if and only if $a' < b'$. d) $a = b$ if and only if $a'' = b''$.
   e) $a = b'$ if and only if $a'' = b$.

7. Let A be a well-ordered class. If $x \in A$, prove that the immediate successor of x and the immediate predecessor of x (if it exists) are unique.
8. Let A be a partially ordered class; prove that B is a section of A if and only if $(B, A - B)$ is a cut of

A.

9. Let A be a well-ordered class; prove the following:

   a) The intersection of any family of sections of A is a section of A.

   b) The union of any family of sections of A is a section of A.

10. Let A be a well-ordered class and let B and C be initial segments of A. Prove that if $B \subset C$, then B is an initial segment of C.

11. Let A be a fully ordered class. Let $B \subseteq A$ and $m \in B$; prove that B has a least element if and only if $S_m \cap B$ has a least element. (Assume $S_m \cap B \neq \varnothing$).

12. Let A be a fully ordered class. Prove that A is well ordered if and only if every initial segment of A is well ordered. [*Hint*: Use the result of Exercise 11, above.]

13. Let A be a well-ordered class. If $a \in A$, let a' designate the immediate successor of a, and let a'' designate the immediate predecessor of a (if it exists). Prove the following:

    a) $a \leqslant b$ if and only if $a' \leqslant b'$. b) $a = b$ if and only if $a' = b'$.
    c) $a < b$ if and only if $a' < b'$. d) $a = b$ if and only if $a'' = b''$.
    e) $a = b'$ if and only if $a'' = b$.

14. Let A be a well-ordered class; if $a \in A$, let a' designate the immediate successor of a. An element q in A will be called a *limit element* of A if q is not the least element of A and q does not have an immediate predecessor. Prove the following:

    a) q is a limit element of A iff $[a < q \Rightarrow a' < q]$.

    b) q is a limit element of A iff $q = \sup\{x \in A : x < q\}$.

6 ISOMORPHISM BETWEEN WELL-ORDERED CLASSES

The purpose of this section is to prove a remarkable property of well-ordered classes: if A and B are any two well-ordered classes, either A is isomorphic with B, or else one of the two is isomorphic with an initial segment of the other. What this means, roughly speaking, is that *well-ordered classes do not differ from one another except in their size*. This fact has many important applications in mathematics, and will be essential to our later discussion of infinite sets and cardinal and ordinal numbers. We begin by proving three preparatory lemmas.

4.58 Lemma Let A be a well-ordered class, and let f be an isomorphism from A to a subclass of A. Then $x \leqslant f(x)$, $\forall x \in A$.

Proof. Assume, on the contrary, that the class $P = \{x \in A : x > f(x)\}$ is nonempty, and let a be the least element of P ; hence, in particular, $f(a) < a$. We now have

$$f(f(a)) < f(a) < a,$$

so $f(a) \in P$, which is impossible because a is the least element of P. Thus $P = \varnothing$, and the lemma is proved. ■

4.59 Lemma Let A be a well-ordered class. There is no isomorphism from A to a subclass of an initial segment of A.

Proof. Assume, on the contrary, that f is an isomorphism from A to a subclass of an initial segment S_a of A. By Lemma 4.58, $a \leqslant f(a)$, so $f(a) \notin S_a$; this is impossible, for the range of f is assumed to be a subclass of S_a. Hence a function f of the kind we assumed cannot exist. ■

4.60 Corollary No well-ordered class is isomorphic with an initial segment of itself.

4.61 Lemma Let A and B be well-ordered classes. If A is isomorphic with an initial segment of B, then B is not isomorphic with any subclass of A.

Proof. Let $f : A \rightarrow S_b$ be an isomorphism from A to an initial segment of B. Assume there exists an isomorphism $g : B \rightarrow C$ where $C \subsetneq A$. Obviously $g : B \rightarrow A$ is a function; $g : B \rightarrow A$ and $f : A \rightarrow S_b$ are both injective and increasing, hence their composite f o $g : B \rightarrow S_b$ is injective and increasing; that is, by 4.48, f o g is an isomorphism from B to its range which is a subclass of S_b. However, by Lemma 4.59, this is impossible; hence the isomorphism g that was assumed cannot exist. ■

The next theorem is widely used in mathematics. It will play a significant role in discussions later in this book, and has important applications.

4.62 Theorem Let A and B be well-ordered classes; exactly one of the following three cases must hold:

i) A is isomorphic with B.

ii) A is isomorphic with an initial segment of B.

iii) B is isomorphic with an initial segment of A.

Proof. We begin by proving that the following holds in any well-ordered class X.

I. Let S_x and S_y be initial segments of X; if $x < y$, then S_x is an initial segment of S_y.

Indeed, if $x < y$, then clearly $S_x \subset S_y$; furthermore, S_x is a section of S_y, for

$$[u \in S_x \text{ and } v \leqslant u] \Rightarrow v \leqslant u < x \Rightarrow v \in S_x;$$

thus by 4.56 (note that $S_x \neq S_y$ because $x \neq y$) we conclude that S_x is an initial segment of S_y.

Now let A and B be well-ordered classes, and let C be the following subclass of A:

$$C = \{x \in A : \exists r \in B \ni S_x \approxeq S_r\}.$$

If $x \in C$, there is no more than one $r \in B$ such that $S_x \approxeq S_r$; for suppose $S_x \approxeq S_r$ and $S_x \approxeq S_t$, where $r \neq t$, say $r < t$. By I, S_r is an initial segment of S_t; but $S_r \approxeq S_x \approxeq S_t$, and this is impossible by 4.60; thus for each $x \in C$, the element $r \in B$ such that $S_x \approxeq S_r$ is unique. Let us designate the unique $r \in B$ corresponding to x by $F(x)$; thus $F : C \rightarrow B$ is a function. In particular, if $D = \operatorname{ran} F$, then $F : C \rightarrow D$ is a function; we will show next that $F : C \rightarrow D$ is an isomorphism.

F is injective. Suppose $F(u) = F(v) = r$, that is, $S_u \approxeq S_r \approxeq S_v$. If $u \neq v$, say $u < v$, then by I, S_u is an initial segment of S_v, and this is impossible by 4.60. We conclude that $u = v$.

F is increasing. Suppose $u \leqslant v$, where $F(u) = r$ and $F(v) = t$; hence $S_u \approxeq S_r$ and $S_v \approxeq S_t$. Assume that $t < r$, hence by I, S_t is an initial segment of S_r; now $S_u \subseteq S_v$, so

a) S_v is isomorphic with an initial segment of S_r, and

b) S_r is isomorphic with a subclass of S_v.

This is impossible by 4.61, hence we conclude that $r \leqslant t$, that is, $F(u) \leqslant F(v)$. It follows, by 4.48, that $F : C \rightarrow D$ is an isomorphism.

Next, we will show that C is a section of A; that is, given $c \in C$ and $x < c$,we will prove that $x \in C$. If $F(c) = r$, then $S_c \approxeq S_r$, that is, there exists an isomorphism $g : S_c \rightarrow S_r$. It is a simple exercise to prove that

$$g_{[S_x]} : S_x \rightarrow S_{g(x)}$$

is an isomorphism; the details are left as an exercise for the reader. Thus $S_x \approxeq S_{g(x)}$, so $x \in C$.

An analogous argument shows that D is a section of B. Thus by 4.56, our theorem will be proven if we can show that the following is *false*:

C is an initial segment of A, and D is an initial segment of B.

Indeed, suppose the above to be true: say $C = S_x$ and $D = S_r$; we have proven that $F : C \rightarrow D$ is an isomorphism, that is, $C \approxeq D$,so $S_x \approxeq S_r$. But then $x \in C$, that is, $x \in S_x$, which is absurd; this proves that one of the conditions (i), (ii) or (iii) necessarily holds. The fact that no two of these conditions holds simultaneously follows from 4.60 and 4.61. ■

4.63 Corollary Let A be a well-ordered class; every subclass of A is isomorphic with A or an initial segment of A.

Proof. If B is a subclass of A, then B is well ordered; hence by 4.62, $B \approxeq A$, or B is isomorphic with an initial segment of A,or A is isomorphic with an initial segment of B. In order to prove our result we must show that the last case cannot hold; indeed, suppose it does: then by 4.61, B is not isomorphic with any subclass of A. But $B \approxeq B$ and B is a subclass of A, so we have a contradiction; thus the last case cannot hold. ■

EXERCISES 4.6

1. In the proof of Theorem 4.62, prove that $g_{[S_x]} : S_x \rightarrow S_{g(x)}$ is an isomorphism.
2. In the proof of Theorem 4.62, prove that D is a section of B.
3. Let A be a well-ordered class. Prove that the identity mapping I_A is the only isomorphism from A to A.
4. Let A and B be well-ordered classes. Prove that if $f : A \rightarrow B$ and $g : B \rightarrow A$ are isomorphisms, then $g = f^{-1}$.

5. Let A and B be well-ordered classes. Prove that there exists at most one isomorphism $f : A \to B$.

6. Let A and B be well-ordered classes. Prove that if A is isomorphic with a subclass of B, and B is isomorphic with a subclass of A, then A is isomorphic with B.

7. Let A and B be well-ordered classes. Prove that if A is isomorphic with a class containing B, and B is isomorphic with a class containing A, then A is isomorphic with B.

8. Let A and B be well-ordered classes. Suppose that A has no greatest element; suppose that every element of B (except the least element) has an immediate predecessor. Prove that B is isomorphic with a section of A.

5

The Axiom of Choice and Related Principles

1 INTRODUCTION

From here on, throughout the remainder of this book, we will be concerned mainly with sets.

In this chapter we will discuss a concept which is one of the most important, and at the same time one of the most controversial, principles of mathematics. In 1904, Zermelo brought attention to an assumption which is used implicitly in a variety of mathematical arguments. This assumption does not follow from any previously known postulates of mathematics or logic, hence it must be taken as a new axiom; Zermelo called it the *Axiom of Choice*. The Axiom of Choice has significant implications in many branches of mathematics, and consequences so powerful as, sometimes, to defy credibility. The controversy over this principle continues in our day; we will present some of its aspects in this chapter.

In order to illustrate where the Axiom of Choice intrudes in common mathematical arguments, let us examine the following statement.

5.1 Let A be a nonempty, partially ordered set, and suppose that there are no maximal elements in A; then there exists a nonterminating, increasing sequence $x_1 < x_2 < x_3 < \cdots$ of elements of A.

Proof. A is nonempty by hypothesis, hence we may select an arbitrary element of A and call it x_1. By induction, suppose that we are given $x_1 < x_2 < \cdots < x_n$; we define A_n to be the set of all the elements $x \in A$ such that $x > x_n$. A_n is nonempty, for if it were empty, then x_n would be maximal, contradicting one of our assumptions. We select an arbitrary element of A_n and call it x_{n+1}; thus we have $x_1 < \cdots < x_n < x_{n+1}$.

This inductive process defines an increasing sequence $S_n = \{x_1, x_2, \ldots, x_n\}$ for each natural number n; that is, it give us

$$S_1 = \{x_1\}, \quad S_2 = \{x_1, x_2\}, \quad S_3 = \{x_1, x_2, x_3\},$$

and so on. Now if we let $S = \bigcup_{n \in N} S_n$ (where N is the set of all the positive integers), then S is the nonterminating sequence $x_1 < x_2 < x_3 < \cdots$ that we are seeking.

A careful examination of the above argument will reveal that we have used an assumption which is by no means self-evident or undisputably plausible. What we have, in fact, done is to assume that we can make an *infinite succession of arbitrary choices*. It is common enough, in mathematics, to make *one* arbitrary choice (we do this every time we can say "let x be an arbitrary element of A"), and experience confirms that we can make a finite succession of choices; but to make an infinite succession of choices is to carry an argument through an infinite number of steps— and nothing in our experience or in the logic we habitually use justifies such a process.

In the proof of 5.1, it was necessary to choose the elements x_1, x_2, x_3, etc., *in succession*, for each choice depended on the preceding ones. The fact that the choices are successive may appear to be the most disturbing element in the whole proof, for this involves a time factor (an infinite *succession* of acts, each on requiring a certain amount of time, would take infinitely long). However, the argument can be altered in such a way that all the choices are made simultaneously and independently of one another; we proceed as follows.

Let us admit that from each nonempty subset $B \subseteq A$ it is possible to choose an arbitrary element $\mathbf{r}_B$, to be called the "representative" of B. Note that in this case each choice is independent of the others; hence, in a manner of speaking, all the choices can be made simultaneously. Returning to the proof of 5.1, if x_n and A_n are given, we may define x_{n+1} to be the representative of A_n; in other words, instead of choosing representatives for A_1, A_2, A_3, etc., in succession, we have chosen representatives for *all* the nonempty subsets of A in advance. (Of course, this requires that we make many more choices than are needed for our original argument, but this is the price we must pay to substitute simultaneous choices for successive ones.)

The preceding paragraph makes it clear that the *successive* nature of the choices is not the crux of the problem; the problem is: *Can we make infinitely many choices*—be they successive or simultaneous?

It is worth noting that in certain particular cases the answer to this question is an obvious "yes." For example, if A is a well-ordered set, we may define the "representative" of each nonempty subset $B \subseteq A$ to be the *least* element of B; because A is well ordered, we have a law at our disposal which *provides us* with a representative for each nonempty subset of A. The situation in 5.1, however, is completely different, for *we do not have any ready-made rule* which is able to furnish us with representatives. It is only the latter case—as in 5.1—which is of interest to us here.

In the proof of 5.1 we speak of "selecting" an element of A_n; clearly, we do not wish to introduce the notion of "selecting" as a new undefined concept of set theory, so we avoid the use of this word by letting an appropriate function "select" representatives.

5.2 Definition Let A be a set; let us agree to write $\mathscr{P}'(A)$ for $\mathscr{P}(A) - \{\emptyset\}$. By a *choice function* for A we mean a function $\mathbf{r} : \mathscr{P}'(A) \to A$ such that

$$\forall B \in \mathscr{P}'(A), \quad \mathbf{r}(B) \in B.$$

We will sometimes write $\mathbf{r}_B$ for $\mathbf{r}(B)$ and call $\mathbf{r}_B$ the *representative* of B.

5.3 Example Let $A = \{a, b, c\}$; an example of a choice function for A is the function $\mathbf{r}$ given in the following table.

B	$\mathbf{r}(B)$
$\{a, b, c\}$	a
$\{a, b\}$	a
$\{a, c\}$	c
$\{b, c\}$	c
$\{a\}$	a
$\{b\}$	b
$\{c\}$	c

In the light of Definition 5.2, the question we have been asking can be expressed as follows: **if A is a set, does there exist a choice function for A?** A crucial comment needs to be made at this point: the proof of 5.1 does not require that we *construct* a choice function, it requires merely that a choice function *exist*! Indeed, if **r** is a choice function for A, then—in the controversial step of the proof—we let $x_{n+1} = \mathbf{r}(A_n)$; if we are assured that **r** *exists*, there is no further difficulty.

The Axiom of Choice asserts that every set has a choice function; its intent is to *state the existence* of a choice function for every set, even where, admittedly, none can be actually constructed. It must be emphasized here that to state the existence of a choice function is quite a different thing from producing one, or even claiming that one can be produced. For we are merely asserting that among *all possible* functions from $\mathscr{P}'(A)$ to A, there is *one at least* which maps every B onto an element $x \in B$.

The essence of the Axiom of Choice is that it is an *existential* statement rather than a *constructive* one. Once again, it states that among all possible functions from $\mathscr{P}'(A)$ to A, there is at least one which satisfies the condition of Definition 5.2; this does not seem grossly unreasonable. The Axiom of Choice makes no claim that a choice function can be constructed; hence, it does not assert that the sequence of choices described in the proof of 5.1 can be effectively carried out—and indeed this is not necessary in order for the proof to work.

Before the reader decides whether or not the Axiom of Choice seems plausible to him, he should examine some of the equivalent propositions which are developed in the next few sections of this chapter. Some of them are very powerful indeed, and a rejection of any one of them would entail a rejection of all of them (including, of course, the Axiom of Choice). It is important to note that what all of these principles have in common is the fact that they are nonconstructive: they assert the existence of mathematical objects which cannot be explicitly produced. It is precisely *this* aspect of the Axiom of Choice and related principles which makes them unacceptable to the intuitionists (see page 17, Section 5, Chapter 0), who claim that mathematical existence and constructibility are one.

Without reentering into the arguments for and against admitting nonconstructive propositions into mathematics, we can say this: intuitively, the Axiom of Choice cannot be rejected outright, nor can we feel truly certain of its validity. A more important consideration, however, is the fact that it has been proven that the Axiom of Choice does not contradict the other axioms of set theory, nor is it a consequence of them. Thus it has the same status as another famous axiom in mathematics, namely Euclid’s “Fifth Postulate.” We can have a “standard” set theory in which we postulate the Axiom of Choice, and “nonstandard” set theories in which we postulate alternatives to the Axiom of Choice.

In conclusion, since the Axiom of Choice is neither a consequence of the other axioms of set theory nor in conflict with them, it is impossible to make a decision *pro* or *con* on purely logical grounds. Since the Axiom of Choice involves an area of mathematics (namely, infinite sets) which is outside the realm of our experience, it will never be possible to confirm it or deny it by “observation.” In the final analysis, the decision must be a purely personal one for each individual mathematician to make; it is a matter of personal taste.

2 THE AXIOM OF CHOICE

In the preceding section, we have given the background for our next axiom:

A10 (*Axiom of Choice*). Every set has a choice function.

In the literature there are several other ways of stating the Axiom of Choice which are equivalent to our Axiom A10. We shall present two of these statements here, and several more in the exercises following this section.

Ch 1 Let $\mathscr{A}$ be a set whose elements are mutually disjoint, nonempty sets. There exists a set C which consists of exactly one element from each $A \in \mathscr{A}$.

If $\mathscr{A}$ is a family of disjoint, nonempty sets, then the set C described in Ch 1 is called a *choice set* for $\mathscr{A}$. Thus, Ch 1 asserts that every set of disjoint, nonempty sets has a choice set.

Ch 2 Let $\{A_i\}_{i \in I}$ be a set of sets. If I is nonempty and each A_i is nonempty, then $\prod_{i \in I} A_i$ is nonempty.

Let us show, first, that A10 ⇒ Ch 1. Suppose $\mathscr{A}$ is a set whose elements are mutually disjoint, nonempty sets, and let

$$A = \bigcup_{X \in \mathscr{A}} X.$$

Clearly, $\mathscr{A} \subseteq \mathscr{P}'(A)$; by A10, there is a function $\mathbf{r} : \mathscr{P}'(A) \to A$ such that $\mathbf{r}(B) \in B$ for each $B \in \mathscr{P}'(A)$; if $C = \bar{\mathbf{r}}(\mathscr{A})$, it follows immediately that C is the set required in Ch. 1.

Next, Ch 1 ⇒ A10. If A is a set and $B \subseteq A$, let $Q_B = \{(B, x) : x \in B\}$. If B and D are distinct, then Q_B and Q_D are disjoint, for Q_B consists of pairs (B, x), whereas Q_D consists of pairs (D, x). Thus the family $\{Q_B\}_{B \in \mathscr{P}'(A)}$ is a set of disjoint, nonempty sets*; it follows by Ch1 that there exists a set C which consists of exactly one element (B, x) from each Q_B; it is easily verified that C is choice function for A.

The fact that A10 ⇒ Ch 2 follows easily from the definition of a product of sets. Indeed, let $\{A_i\}_{i \in I}$ be a set of nonempty sets, and let

$$A = \bigcup_{i \in I} A_i;$$

by A10, there exists a function $\mathbf{r} : \mathscr{P}'(A) \to A$ such that $\mathbf{r}(B) \in B$ for each $B \in \mathscr{P}'(A)$; hence, in particular, $\mathbf{r}(A_i) \in A_i$ for each $i \in I$. If we define $\mathbf{a}$ by $\mathbf{a}(i) = \mathbf{r}(A_i)$, then $\mathbf{a}$ is a function from I to A such that $\mathbf{a}(i) \in A_i$ for each $i \in I$; that is, $a \in \prod_{i \in I} A_i$. Thus $\prod_{i \in I} A_i$ is nonempty.

The fact that Ch 2 ⇒ A10 can be proven by an argument similar to the above; the details are left to the reader.

In Chapter 2 we promised to give a characterization of surjective functions whose proof depends on the Axiom of Choice.

5.4 Theorem Let A be a set and let $f : A \to B$ be a function; $f : A \to B$ is surjective if and only if there exists a function $g : B \to A \ni f \circ g = I_B$.

Proof

i) Suppose there exists a function $g : B \to A$ such that $f \circ g = I_B$; the proof that $f : A \to B$ is surjective is given in part (ii) of the proof of 2.24.

ii) Conversely, suppose that $f : A \to B$ is surjective; for each $y \in B$, $\check{f}(y)$ is a nonempty subset of A. If $\mathbf{r}$ is a choice function for A, we define $g : B \to A$ by $g(y) = \mathbf{r}[\check{f}(y)]$, $\forall y \in B$. In simple terms, for each $y \in B$ we let $g(y)$ be an arbitrary element of $\check{f}(y)$. It is obvious that if $x = g(y)$, then $x \in \check{f}(y)$, hence $f(x) = y$; thus

$$[f \circ g](y) = f(x) = y = I_B(y). \ \blacksquare$$

For the sake of simplicity we have proven Theorem 5.4 in the case where A is a set; using a slightly stronger form of the Axiom of Choice we can prove 5.4 in the more general case where A is any class; we omit the details.

EXERCISES 5.2

1. Let A be a set and let $f : A \to B$ be a surjective function. Prove that there exists a subset $C \subseteq A$ such that C is in one-to-one correspondence with B. [*Hint*: Use 5.4.]
2. Let A be a set, let $f : B \to C$ and $g : A \to C$ be functions, and suppose that ran$f \subseteq$ ran$g$. Prove that there exists a function $h: B \to A$ such that $g \circ h = f$. [*Hint*: Use the Axiom of Choice.]
3. Let $\{A_i\}_{i\in I}$ be an indexed family of classes, where I is a set. Prove that there exists $J \subseteq I$ such that

$$\{A_i : i \in I\} = \{A_j : j \in J\}$$

and, in $\{A_j\}_{j\in J}$, each A_j is indexed only once (that is, $A_i = A_j \Rightarrow i = j$). [*Hint*: Use Remark 2.38 and the Axiom of Choice.]

4. Prove that the statement of Theorem 5.4 implies the Axiom of Choice.

In each of the following problems a proposition is stated. Prove that this proposition is equivalent to the Axiom of Choice.

5. Let $\mathscr{A}$ be a set of disjoint, nonempty sets. There exists a function f, whose domain in $\mathscr{A}$, such that for all $A \in \mathscr{A}$, $f(A) \in A$.
6. Let E be a set and suppose $G \subseteq E \times E$. Let $A =$ dom G and $B =$ ran G; then there exists a function $f : A \to B$ such that $f \subseteq G$.
7. Let $\mathscr{A}$ be a set whose elements are nonempty sets, and let $A = \bigcup_{X\in\mathscr{A}} X$. Then, corresponding to every function g: $\mathscr{A} \to \mathscr{A}$, there exists a function g*: $\mathscr{A} \to A$ such that $g^*(B) \in g(B)$.
8. Let B be a set and let $f : A \to B$ be a function; then there exist subsets $C \subseteq A$ and $g \subseteq f$ such that $g : C \to B$ is an injective function and ran$g =$ ran$f$.

3 AN APPLICATION OF THE AXIOM OF CHOICE

The purpose of this section is to develop a consequence of the Axiom of Choice. The result we are about to prove is valuable as a stepping stone which will enable us to prove the important maximal principles that follow in the next section.

Let A be a partially ordered set such that every chain of A has a sup in A; assume that A has a least element p. We intend to show that there exists an element $a \in A$ such that a has no immediate successor.

In order to show this, we will suppose that every element $x \in A$ has an immediate successor; this assumption will lead to a contradiction.

If every element of A has an immediate successor, then we can define a function $f : A \to A$ such that for each $x \in A$, $f(x)$ is an immediate successor of x. Indeed, let T_x be the set of all the immediate successors of x; by the Axiom of Choice, there exists a choice function g such that $g(T_x) \in T_x$. We define f by letting $f(x) = g(T_x)$; clearly, $f(x)$ is an immediate successor of x.

5.5 Definition A subset $B \subseteq A$ is called a *p-sequence* if the following conditions are satisfied.

α) $p \in B$,

β) if $x \in B$, then $f(x) \in B$,

γ) if C is a chain of B, then sup $C \in B$.

There *are* p-sequences; for example, A is a p-sequence.

5.6 Lemma Any intersection of p-sequences is a p-sequence.

The proof is left as an exercise for the reader.

Let P be the intersection of all the p-sequences. (Note that $P \neq \emptyset$ because $p \in P$). By 5.6, P is a p-sequence.

5.7 Definition An element $x \in P$ is called *select* if it is comparable with every element $y \in P$.

5.8 Lemma Suppose x is select, $y \in P$, and $y < x$. Then $f(y) \leqslant x$.

Proof. $y \in P$, P is a p-sequence, hence by (β), $f(y) \in P$. Now, x is select, so either $f(y) \leqslant x$ or $x < f(y)$. By hypothesis $y < x$;so if $x < f(y)$,we have $y < x < f(y)$, which contradicts the assertion that $f(y)$ is the immediate successor of y. Hence $f(y) \leqslant x$. ■

5.9 Lemma Suppose x is select. Let

$$B_x = \{y \in P : y \leq x \text{ or } y \geq f(x)\}.$$

Then B_x is a p-sequence.

Proof. We will show that B_x satisfies the three conditions which define a p-sequence.

α) Since p is the least element of A, $p \leqslant x$, hence $p \in B_x$.

β) Suppose $y \in B_x$; then $y \leqslant x$ or $y \geq f(x)$. Consider three cases:

1) $y < x$. Then $f(y) \leqslant x$ by 5.8, hence $f(y) \in B_x$.

2) $y = x$. Then $f(y) = f(x)$, thus $f(y) \geq f(x)$; hence $f(y) \in B_x$.

3) $y \geqslant f(x)$. But $f(y)>y$,so $f(y)>f(x)$; hence $f(y) \in B_x$.

In each case we conclude that $f(y) \in B_x$.

γ) If C is a chain of B_x, let $m =$ sup C. For each $y \in B_x$,$y \leqslant x$ or $y \geq f(x)$. If $\exists y \in C$ $y \geq f(x)$, then (since $m \geq y$) $m \geq f(x)$,so $m \in B_x$. Otherwise, $\forall y \in C$, $y \leqslant x$; thus x is an upper bound of C,so $m \leqslant x$. Thus again $m \in B_x$. ■

5.10 Corollary If x is select, then $\forall y \in P, y \leqslant x$ or $y \geq f(x)$.

Proof. B_x is a p-sequence; P is the intersection of all p-sequences; hence $P \subseteq B_x$. But $B_x \subseteq P$ by definition, hence $P = B_x$. So $\forall y \in P, y \leqslant x$ or $y \geq f(x)$. ■

5.11 Lemma The set of all select elements is a p-sequence.

Proof

α) p is select because it is less than (hence comparable to) each $y \in P$.

β) Suppose x is select, by 5.10. $\forall y \in P$, either $y \leqslant x$ (in which case $y \leqslant f(x)$ because $x < f(x)$) or $y \geq f(x)$. Thus $f(x)$ is select.

γ) Let C be a chain of select elements and let $m = \sup C$; let $y \in P$. If $\exists x \in C\, y \leqslant x$, then $y \leqslant m$ (because $x \leqslant m$). Otherwise, $\forall x \in C, x \leqslant y$, hence y is an upper bound of C,so $m \leqslant y$. Thus m is select. ■

5.12 Corollary P is fully ordered.

Proof. The set S of all the select elements is a p-sequence; P is the intersection of all the p-sequences; hence $P \subseteq S$. But $S \subseteq P$ (by definition a select element is in P), so $P = S$. Thus each element of P is select, that is, is comparable to each element of P. ■

Corollary 5.12 produces a contradiction. Indeed, let $m = \sup P$; by condition *(γ), $m \in P$* because P is a chain of P. But by condition *(β), $f(m) \in P$*, hence $f(m) \leqslant m$; this contradicts the assertion that $f(m)$ is an immediate successor of m. We conclude:

5.13 Theorem Let A be a partially ordered set such that (1) A has a least element p and (2) every chain of A has a sup in A. Then there is an element $x \in A$ which has no immediate successor.

4 MAXIMAL PRINCIPLES

The propositions we are about to develop are widely used in mathematics to prove theorems by "nonconstructive" methods. They assert the existence of mathematical objects which cannot be constructed. For example, in linear algebra a maximal principle can be used to prove that every vector space has a basis, although, in general, it is impossible to exhibit such a basis.

The maximal principles are consequences of the Axiom of Choice. Furthermore, as we shall verify in Section 6, they are equivalent to the Axiom of Choice.

5.14 Theorem (*Hausdorff's Maximal Principle*). Every partially ordered set has a maximal chain.

Proof. Let A be a partially ordered set, and let $\mathscr{S}$ be the set of all the chains of A, ordered by inclusion. $\mathscr{S}$ has a least element, namely the empty set. Now let $\mathscr{C}$ be a chain of $\mathscr{S}$ and let

$$K = \bigcup_{C \in \mathscr{C}} C;$$

we will show that $K \in \mathscr{S}$. Indeed, if $x, y \in K$, then $x \in D$ and $y \in E$ for some elements $D \in \mathscr{C}$ and $E \in$

$\mathscr{C}$; but $\mathscr{C}$ is a chain of $\mathscr{S}$, hence $E \subseteq D$ or $D \subseteq E$, say $E \subseteq D$; thus $x, y \in D$. But D is a chain of A (remember that $\mathscr{S}$ is the set of all the chains of A), so x and y are comparable; this proves that K is a chain of A, that is, $K \in \mathscr{S}$. By 4.26 $K = \sup \mathscr{C}$; it follows that the conditions of Theorem 5.13 are satisfied by $\mathscr{S}$. Thus, by 5.13, there is an element $C \in \mathscr{S}$ which has no immediate successor; that is, there exists no $x \in A - C$ such that $C \cup \{x\}$ is a chain of A. Thus, clearly, C is a maximal chain. ■

5.15 Definition A partially ordered set A is said to be *inductive* if every chain of A has an upper bound in A.

5.16 Theorem (*Zorn's Lemma*). Every inductive set has at least one maximal element.

Proof. Let A be an inductive set; by 5.14, A has a maximal chain C; by 5.15, C has an upper bound m. Now suppose there exists an element $x \in A$ $x > m$; then $x \notin C$, but x is comparable with (to be exact, x is greater than) every element of C. Thus, $C \cup \{x\}$ is a chain, contradicting the assertion that C is a maximal chain; hence there exists no element $x \in A$ such that $x > m$,so m is a maximal element of A. ■

Theorems 5.14 and 5.16 can be stated in a somewhat stronger form as follows:

5.17 Theorem Every partially ordered set has a maximal well-ordered subset.

5.18 Theorem (Let us call a partially ordered set A *weakly inductive* if every well-ordered subset of A has an upper bound in A.) Every weakly inductive set has at least one maximal element.

The proofs of the last two theorems are similar to those of Theorems 5.14 and 5.16; they are left as an exercise for the reader.

EXERCISES 5.4

1. Let A be a partially ordered set and let $\mathscr{A}$ be the set of all the well-ordered subsets of A. For $C \in \mathscr{A}$ and $D \in \mathscr{A}$, define $C \preccurlyeq D$ if and only if $\preccurlyeq$ is a section of D.

   a) Prove that $\preccurlyeq$ is a partial order relation in $\mathscr{A}$.

   b) Prove that $\mathscr{A}$, ordered by $\preccurlyeq$, is inductive.

   c) Using part (b) and Zorn's Lemma, prove Theorem 5.17.
2. Use the result of Exercise 1, above, to prove Theorem 5.18.
3. Derive Hausdorff's Maximal Principle from Zorn's Lemma.
4. Prove that Zorn's Lemma is equivalent to the following: Let A be an inductive set and let $a \in A$; then A has at least one maximal element b such that $b \geqslant a$.
5. Prove that Hausdorff's Maximal Principle is equivalent to the following: If A is a partially ordered set and B is a chain of A, then A has a maximal chain C such that $B \subseteq C$.
6. Let A be any set with more than one element. Prove that there exists a bijective function $f : A \to A$ such that $f(x) = x$, $\forall x \in A$.
7. A set of sets $\mathscr{A}$ is said to be disjointed if $\forall C, D \in \mathscr{A}$, $C \cap D = \varnothing$. Let $\mathscr{F}$ be a set of sets; prove that $\mathscr{F}$ has a maximal disjointed subset.
8. Let A be a set and let $\mathscr{A}$ be a set of subsets of A; let $\mathscr{A}$ have the following property: $B \in \mathscr{A}$ iff every

finite subset of B belongs to $\mathscr{A}$; then $\mathscr{A}$ is said to be of *finite character*. Let $\mathscr{A}$ be ordered by inclusion and suppose $\mathscr{A}$ is of finite character.

a) Prove that $\mathscr{A}$ is an inductive set.

b) Prove that $\mathscr{A}$ has a maximal element.

9. Prove that every vector space V has a basis. [*Hint*: Consider the set $\mathscr{A}$ of all the linearly independent subsets of V. Use Zorn's Lemma: it is easily verified that any maximal linearly independent subset of V is a basis of V.]

10. Let G be a group and let A be an arbitrary subset of G such that A includes the identity element of G. Prove that among the subgroups of G which are subsets of A, there is a maximal one.

5 THE WELL-ORDERING THEOREM

The well-ordering theorem, which will be presented in this section, is one of the most important consequences of the Axiom of Choice and is an outstanding example of a nonconstructive proposition. It asserts that any set can be well ordered; that is, if A is any set, there exists an order relation G such that A, ordered by G, is a well-ordered set. The proof of the well-ordering theorem gives no indication how such a well-ordering of the elements of A is to be accomplished; it asserts merely that a well-ordering exists.

A finite set A can obviously be well ordered; for example, if $A = \{a, b, c\}$, then $a < b < c$, $b < a < c$, $c < b < a$ are different well-orderings of A. However, no method has yet been discovered to well-order sets such as the set $\mathbb{R}$ of the real numbers; in fact, in the opinion of most mathematicians, it is impossible to construct a well-ordering of $\mathbb{R}$. (This would mean giving a rule for rearranging the elements of $\mathbb{R}$ in such a way that $\mathbb{R}$ would thereby become a well-ordered set.)

Once again, the well-ordering theorem does *not* assert that every set can be *effectively* well ordered; it merely states that, among all possible graphs $G \subseteq A \times A$, there exists one at least which is an order relation which well-orders A.

Let us now prove the well-ordering theorem; note that the proof relies heavily on the Axiom of Choice.

Let A be an arbitrary set. We will consider pairs (B, G), where B is a subset of A, and G is an order relation in B which well-orders B.

Let $\mathscr{A}$ be the family of all such pairs (B, G). We introduce the symbol $\prec$ and define $(B, G) \prec (B',G')$ if and only if

5.19 a) $B \subseteq B'$, b) $G \subseteq G'$, c) $x \in B$ and $y \in B' - B \Rightarrow (x, y) \in G'$.

(Note that the last condition asserts, roughly, that all the elements of B precede all the elements of $B' - B$.) It is easy to verify that $\prec$ is an order relation $\mathscr{A}$; the details are left as an exercise for the reader.

5.20 Lemma Let

$$\mathscr{C} = \{(B_i, G_i)\}_{i \in I}$$

be a chain of $\mathscr{A}$; let

$$B = \bigcup_{i \in I} B_i \quad \text{and} \quad G = \bigcup_{i \in I} G_i .$$

Then $(B, G) \in \mathscr{A}$.

Proof. By 1.40(i), $B \subseteq A$; thus, our result will be established if we can show that G well-orders B. First we verify that G is an order relation in B.

Reflexive. $x \in B \Rightarrow x \in B_i$ for some $i \in I \Rightarrow (x, x) \in G_i \subseteq G$; thus G is reflexive.

Antisymmetric. $(x, y) \in G$ and $(y, x) \in G \Rightarrow (x, y) \in G_i$ and $(y, x) \in G_j$ for some $i \in I$ and $j \in I$; but $\mathscr{C}$ is a chain of $\mathscr{A}$, so $G_i \subseteq G_j$ or $G_j \subseteq G_i$, say $G_i \subseteq G_j$. Thus $(x, y) \in G_j$ and $(y, x) \in G_j$;but G_j is an order relation, so $x = y$. This proves that G is antisymmetric.

Transitive. $(x, y) \in G$ and $(y, z) \in G \Rightarrow (x, y) \in G_i$ and $(y, z) \in G_j$ for some $i \in I$ and $j \in I$; but $\mathscr{C}$ is a chain, so $G_i \subseteq G_j$ or $G_j \subseteq G_i$, say $G_i \subseteq G_j$. Then $(x, y) \in G_j$ and $(y, z) \in G_j$,so $(x, z) \in G_j \subseteq G$. Thus G is transitive.

Now we must show that B is well-ordered by G. Suppose that $D \neq \emptyset$ and $D \subseteq B$; then $D \cap B_i \neq \emptyset$ for some $i \in I$. Now $D \cap B_i \subseteq B_i$, hence $D \cap B_i$ has a least element b in (B_i,G_i); that is $\forall y \in D \cap B_i$, $(b, y) \in G_i$. We will proceed to show that b is the least element of D in (B, G); that is, $\forall x \in D$, $(b, x) \in G$.

Indeed, let $x \in D$: if $x \in B_i$, then $(b, x) \in G_i \subseteq G$. Now suppose $x \notin B_i$; in this case, $x \in B_j$ for some $j \in I$; $B_j \nsubseteq B_i$ because $x \in B_j$ and $x \notin B_i$, hence $(B_j, G_j) \nprec (B_i,G_i)$; it follows that $(B_i,G_i) \prec (B_j, G_j)$. Now we have $b \in B_i$, $x \in (B_j - B_i)$, and $(B_i,G_i) \prec (B_j, G_j)$; thus, by 5.19(c), $(b, x) \in G_j \subseteq G$. This proves that b is the least element of D in (B, G). ■

5.21 Lemma If $\mathscr{C}$, B, and G are defined as above, (B, G) is an upper bound of $\mathscr{C}$.

Proof. Let $(B_i,G_i) \in \mathscr{C}$; clearly $B_i \subseteq B$ and $G_i \subseteq G$. Now suppose that $x \in B_i$, $y \in B$, and $y \notin B_i$; certainly $y \in B_j$ for some $j \in I$. Now $B_j \nsubseteq B_i$ because $y \in B_j$ and $y \notin B_i$,so $(B_j, G_j) \nprec (B_i,G_i)$, hence $(B_i,G_i) \prec (B_j, G_j)$. Now $x \in B_i$ and $y \in (B_j - B_i)$, so by 5.19(c), $(x, y) \in G_j \subseteq G$. Thus $(B_i,G_i) \prec (B, G)$. ■

5.22 Theorem (*Well-ordering Theorem*). Any set A can be well ordered.

Proof. By lemma 5.20 and 5.21, we can apply Zorn's Lemma to $\mathscr{A}$; thus $\mathscr{A}$ has a maximal element (B, G). We will show that $B = A$; hence A can be well-ordered. Otherwise, $\exists x \in (A - B)$; by defining x to be greater than each element of B, we get an extension G^* of G that well-orders $B \cup \{x\}$. (More explicitly, $G^* = G \cup \{(a, x) : a \in B\}$.) This is a contradiction, since (B, G) was assumed to be maximal. ■

6 CONCLUSION

It is clear that the Axiom of Choice can be derived from the well-ordering theorem. Indeed, let A be any set; by the well-ordering theorem, A can be well ordered; if B is a nonempty subset of A, let $f(B)$ be the least element of B. Then f is a choice function on A.

We have now proven the following implications:

Axiom of Choice ⇒ Hausdorff’s Maximal Principle

⇒Zorn’s Lemma ⇒ well-ordering theorem ⇒ Axiom of Choice.

Thus we have established the complete equivalence of the above four propositions.

In the remainder of this book we will accept the Axiom of Choice as one of the axioms of set theory. Thus we will feel free to use the Axiom of Choice in all of our arguments, except if we make an explicit statement to the contrary.

* Note that for each $B \in \mathscr{P}'(A)$, $Q_B \subseteq \mathscr{P}(A) \times A$; thus

$$\{Q_B\}_{B \in \mathscr{P}'(A)} \subseteq \mathscr{P}[\mathscr{P}(A) \times A];$$

hence by A3, and A7, $\{Q_B\}_{B \in \mathscr{P}'(A)}$ is a set.

6

The Natural Numbers

1 INTRODUCTION

Probably the most fundamental—and the most primitive—of all mathematical concepts is that of natural number. The natural, or "counting," numbers are the first mathematical abstraction which we learn as children; every human society—even the most backward and remote—possesses a system of some kind for counting objects.

We all have a clear intuitive understanding of what the natural numbers are: they are 0, 1, 2, 3, and so forth. But his intuitive perception—no matter how clear and immediate it may be—is not sufficient for the purposes of mathematics; in order to do mathematics with numbers, we must articulate this vague perception and transform it into a precise definition. The definition must, of course, faithfully reflect the intuitive notion from which it sprang.

It is our aim in this section to *define* the natural numbers; to be explicit, we will construct a set of objects to be called " natural numbers," and these " natural numbers" will be endowed with all of the properties which are associated with the natural numbers in our mind.

We should carefully note two important requirements of our definition. In the first place, we would like to present the natural numbers without introducing any new undefined notions. We were faced with a similar problem when we were about to introduce the notion of function; we solved it by defining a function to be a certain kind of class (specifically, a class of ordered pairs). Following this example, it is clear that we ought to define natural numbers to be classes (more specifically, *sets*) of some kind; in fact, for each *n* we will define "*n*" to be a set which (intuitively) has *n* elements. Secondly, each natural number must be uniquely defined; that is, for each *n* there must be just one, *unique* object which can be recognized as the "natural number *n*." Thus we must devise a means of fixing exactly one set, among all the sets with *n* elements, and calling this set "*n*."

The numbers "0," "1," "2," etc., which we are about to define will serve as standards in much the same way that the standard yard in Washington, D.C., serves as a norm for measuring length. It matters little whether the standard yard is made of platinum or stainless steel, or whether it is decorated with figures of dancing mermaids; its only use is as a standard of reference, so anything having the same length is, by definition, one yard long. Analogously, it does not matter too much which specific sets we define "0," "1." "2," etc., to be; they will be used as standards of reference, so a set will be said to "have *n* elements" if it is in one-to-one correspondence with the natural number "*n*."

We will proceed as follows: we define

$$0 = \emptyset.$$

In order to define "1," we must fix a set with exactly one element; thus

$$1 = \{0\}.$$

Continuing in this fashion, we define

$$
\begin{aligned}
2 &= \{0, 1\}, \\
3 &= \{0, 1, 2\}, \\
4 &= \{0, 1, 2, 3\}, \quad \text{etc.}
\end{aligned}
$$

The reader should note that 0 = Ø, 1 ={Ø }, 2 ={Ø, {Ø }}, 3 ={Ø, {Ø }, {Ø, {Ø }}}, etc. Our natural numbers are constructions beginning with the empty set.

In view of what has just been indicated, the following also are true:

$$
\begin{aligned}
1 &= \{0\} \\
2 &= 1 \cup \{1\} \\
3 &= 2 \cup \{2\} \quad \text{etc.}
\end{aligned}
$$

This should explain the following definition:

The preceding definitions can be restated, a little more precisely, as follows. If *A* is a set, we define the *successor of A* to be the set A^+, given by

$$A^+ = A \cup \{A\}.$$

Thus, A^+ is obtained by adjoining to *A* exactly one new element, namely the element *A*. Now we define

$$
\begin{aligned}
0 &= \varnothing, \\
1 &= 0^+, \\
2 &= 1^+, \\
3 &= 2^+, \quad \text{etc.}
\end{aligned}
$$

It is clear that these definitions coincide with those given in the preceding paragraph.

From these definitions, it is worth noting that 0 ⊂ 1 ⊂ 2 ⊂ 3 ⊂ ..., and likewise 0 ∈ 1 ∈ 2 ∈ 3 ∈ ....

We have just outlined a method for producing sets which we call “natural number;” beginning with the empty set, we have given directions for constructing, successively, 0 = Ø, 1 = 0^+, 2 = 1^+, and so forth. An important question now is the following: Is there such a thing as the *set of all the natural numbers* (or even the *class of all the natural numbers*)? That is, is there a set (or a class) which contains Ø, and which contains X^+ whenever it contains *X*? Certainly, our method does not enable us to construct it—we are merely given instructions for producing numbers 0, 1, 2, ..., *n up to any n*. Thus we cannot yet speak of the set (or class) of all the natural numbers.

A set *A* is called a *successor set* if it has the following properties:

i) Ø ∈ *A*.

ii) if *X* ∈ *A*, then X^+ ∈ *A*.

It is clear that any successor set necessarily includes all the natural numbers. Motivated by this observation, we introduce the following important axiom.

A11 (*Axiom of Infinity*). There exists a successor set.

As we have noted, every successor set includes all the natural numbers; thus it would make sense to define the “set of the natural numbers” to be the smallest successor set. Now it is easy to verify that any intersection of successor sets is a successor set; in particular, the intersection of all the successor sets is a successor set (it is obviously the smallest successor set). Thus, we are led naturally to the following definition.

6.1 Definition By the *set of the natural numbers* we mean the intersection of all the successor sets. The set of the natural numbers is designated by the symbol ω; every element of ω is called a *natural number*.

2 ELEMENTARY PROPERTIES OF THE NATURAL NUMBERS

In this section we will show that the natural numbers, as we have just defined them, satisfy the five conditions commonly known as the Peano axioms. This set of conditions—it is well known—is another way of defining and characterizing the natural numbers.

6.2 Theorem For each $n \in \omega$, $n^+ \neq 0$.

Proof. By definition, $n^+ = n \cup \{n\}$; thus $n \in n^+$ for each natural number n; but 0 is the empty set, hence 0 cannot be n^+ for any n. ■

6.3 Theorem (*Mathematical Induction*). Let X be a subset of ω; suppose X has the following properties:

i) $0 \in X$.

ii) If $n \in X$, then $n^+ \in X$.

Then $X = \omega$.

Proof. Conditions (i) and (ii) imply that X is a successor set. By 6.1, ω is a subset of every successor set; thus $\omega \subseteq X$. But $X \subseteq \omega$; so $X = \omega$. ■

6.4 Lemma Let m and n be natural numbers; if $m \in n^+$, then $m \in n$ or $m = n$.

Proof. By definition, $n^+ = n \cup \{n\}$; thus, if $m \in n^+$, then $m \in n$ or $m \in \{n\}$; but $\{n\}$ is a singleton, so $m \in \{n\}$ iff $m = n$. ■

6.5 Definition A set A is called *transitive* if, for each $x \in A$, $x \subseteq A$.

For example, the number 3 is a transitive set; indeed, its elements are 0, 1, 2, that is, $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$. It is clear that each of these elements is a subset of 3. The same is true of every natural number, as we shall prove next.

6.6 Lemma Every natural number is a transitive set.

Proof. Let X be the set of all the elements of ω which are transitive sets; we will prove, using mathematical induction (Theorem 6.3), that $X = \omega$; it will follow that every natural number is a transitive set.

i) $0 \in X$, for if 0 were not a transitive set, this would mean that $\exists\, y \in 0$ such that y is not a subset of 0; but this is absurd, since $0 = \varnothing$.

ii) Now suppose that $n \in X$; we will show that $n^+ \in X$; that is, assuming that n is a transitive set, we will show that n^+ is a transitive set. Let $m \in n^+$; by 6.4, $m \in n$ or $m = n$. If $m \in n$, then (because n is transitive) $m \subseteq n$; but $n \subseteq n^+$, so $m \subseteq n^+$. If $m = n$, then (because $n \subseteq n^+$) $m \subseteq n^+$; thus in either case, $m \subseteq n^+$, so $n^+ \in X$. It follows by 6.3 that $X = \omega$. ■

6.7 Theorem Let n and m be natural numbers. If $n^+ = m^+$, then $n = m$.

Proof. Suppose $n^+ = m^+$; now $n \in n^+$, hence $n \in m^+$; thus, by 6.4, $n \in m$ or $n = m$. By the very same argument, $m \in n$ or $m = n$. If $n = m$, the theorem is proved. Now suppose $n \neq m$; then $n \in m$ and $m \in n$. Thus, by 6.5 and 6.6, $n \subseteq m$ and $m \subseteq n$, hence $n = m$. ■

The Peano axioms for the natural numbers are:

P1 $0 \in \omega$.

P2 If $n \in \omega$, then $n^+ \in \omega$.

P3 For each $n \in \omega$, $n^+ \neq 0$.

P4 If X is a subset of ω such that

i) $0 \in X$, and

ii) if $n \in X$, then $n^+ \in X$, then $X = \omega$.

P5 If $n, m \in \omega$ and $n^+ = m^+$, then $n = m$.

P1 and P2 follow immediately from our definition of ω. P3 is given by Theorem 6.2, P4 is given by Theorem 6.3, and P5 is given by Theorem 6.7. Thus our set ω satisfies the Peano axioms.

EXERCISES 6.2

1. Prove that A is a transitive set if and only if the following holds: If $B \in C$ and $C \in A$, then $B \in A$.
2. Prove that if A and B are transitive sets, then $A \cup B$ and $A \cap B$ are transitive sets.
3. Let A and B be sets. Prove that if $A = B$, then $A^+ = B^+$.
4. Use 6.3, to prove that for every natural number n, $n \notin n$.
5. Prove the following, where $m, n, p \in \omega$.

a) $n \neq n^+$. b) If $m \in n$, then $n \notin m$.

c) If $n \in m$ and $m \in p$, then $n \in p$. d) If $m \in n$, then $m^+ \subseteq n$.

6. a) Prove by induction: If $A \in n$ and $n \in \omega$, then $A \in \omega$. Conclude that ω is a transitive set.

b) Prove that if $A^+ \in \omega$, then $A \in \omega$.

7. Prove that no natural number is a successor set.
8. Prove that no natural number is a subset of any of its elements.
9. Prove by induction: If $n \in \omega$, then either $n = 0$ or $n = m^+$ for some $m \in \omega$.
10. Let $n \in \omega$. Prove the following.

a) $\cup\, n^+ = n$.

b) $\cup\, \omega = \omega$ (see Remark 1.47).

11. Let A be a nonempty subset of ω. Prove that if $\cup\, A = A$ then $A = \omega$.

3 FINITE RECURSION

Induction is commonly used not only as a method of proof but also as a method of definition. For example, a familiar way of introducing exponents in arithmetic is by means of the “inductive definition”

I. $a^0 = 1$,

II. $a^{n+1} = a^n a, \forall\, n \in \omega$.

The pair of Conditions I and II is meant to be interpreted as a rule which specifies the meaning of a^n for each natural number n. Thus, by I, $a^0 = 1$; by I and II,

$$a^1 = a^{0+1} = a^0 a = 1a = a;$$

by II again,

$$a^2 = a^{1+1} = a^1 a = aa;$$

continuing in this fashion—using Condition II repeatedly—the numbers $a^0, a^1, a^2, \ldots, a^n$ are defined in succession up to any chosen n.

Inductive definitions such as the one we have just seen abound in mathematics. The situation, in almost every case, is the following: We have a set A, a function $f : A \to A$, and a fixed element $c \in A$. We define a function $\gamma : \omega \to A$ by means of the two Conditions.

I. $\gamma\,(0) = c$,

II. $\gamma\,(n^+) = f\,(\gamma\,(n)), \forall n \in \omega$

The reader should recognize that exactly this situation prevails in our preceding example. In that example, A is the set of the real numbers, c is 1, $\gamma\,(n)$ is denoted by a^n, and f is the function defined by $f\,(x) = xa$.

For another example, let $\mathbb{R}$ be the set of the real numbers and let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f\,(x) = x^2$. We define a function $\gamma : \omega \to \mathbb{R}$ by

I. $\gamma\,(0) = 2$,

II. $\gamma\,(n + 1) = f\,(\gamma\,(n)) =[\gamma\,(n)]^2$.

The reader will recognize this as an inductive definition of the function $\gamma\,(n) = 2^{2n}$.

In each of the foregoing examples, it is reasonable to believe that if γ exists, then Conditions I and II determine the values $\gamma\,(n)$ for every $n \in \omega$. However, Conditions I and II are insufficient in themselves to guarantee that a function such as γ exists. If we are to accept definition by induction as a legitimate way of constructing mathematical objects, then we must first establish the fact that γ —the function which we purport to be defining—actually exists and is uniquely determined. The purpose of the following theorem is to perform this important task.

6.8 Recursion Theorem Let A be a set, c a fixed element of A, and f a function from A to A. Then there exists a unique function $\gamma : \omega \to A$ such that

I. $\gamma(0) = c$, and

II. $\gamma(n^+) = f(\gamma(n))$, $\forall\, n \in \omega$.

Proof. First, we will establish the *existence* of γ. It should be carefully noted that γ is a set of ordered pairs which is a function and satisfies Conditions I and II. More specifically, γ is a subset of $\omega \times A$ with the following four properties:

1) $\forall n \in \omega, \exists x \in A \ni (n, x) \in \gamma$.
2) If $(n, x_1) \in \gamma$, and $(n, x_2) \in \gamma$, then $x_1 = x_2$.
3) $(0, c) \in \gamma$.
4) If $(n, x) \in \gamma$, then $(n^+, f(x)) \in \gamma$.

Properties (1) and (2) express the fact that γ is a function from ω to A, while properties (3) and (4) are clearly equivalent to I and II. We will now construct a graph γ with these four properties.

Let

$$\mathscr{A} = \{G : G \subseteq \omega \times A \text{ and } G \text{ satisfies (3) and (4)}\};$$

$\mathscr{A}$ is nonempty, because $\omega \times A \in \mathscr{A}$. It is easy to see that any intersection of elements of $\mathscr{A}$ is an element of $\mathscr{A}$; in particular,

$$\gamma = \bigcap_{G \in \mathscr{A}} G$$

is an element of $\mathscr{A}$. We proceed to show that γ is the function we require.

By construction, γ satisfies (3) and (4), so it remains only to show that (1) and (2) hold.

1) It will be shown by induction that dom $\gamma = \omega$, which clearly implies (1). By (3), $(0, c) \in \gamma$, so $0 \in$ dom γ; now suppose $n \in$ dom γ. Then $\exists\, x \in A\ (n, x) \in \gamma$; by (4), then, $(n^+, f(x)) \in \gamma$, so $n^+ \in$ dom γ. Thus, by Theorem 6.3, dom $\gamma = \omega$.

2) Let

$$N = \{n \in \omega : (n, x) \in \gamma \text{ for no more than one } x \in A\}.$$

It will be shown by induction that $N = \omega$. To prove that $0 \in N$, we first assume the contrary; that is, we assume that $(0, c) \in \gamma$ and $(0, d) \in \gamma$ where $c \neq d$. Let $\gamma^* = \gamma - \{(0, d)\}$; certainly γ^* satisfies (3); to show that γ^* satisfies (4), suppose that $(n, x) \in \gamma^*$. Then $(n, x) \in \gamma$, so $(n^+, f(x)) \in \gamma$; but $n^+ \neq 0$ (Theorem 6.2), so $(n^+, f(x)) \neq (0, d)$, and consequently $(n^+, f(x)) \in \gamma^*$. We conclude that γ^* satisfies (4), so $\gamma^* \in \mathscr{A}$; but γ is the intersection of all the elements of $\mathscr{A}$, so $\gamma \subseteq \gamma^*$. This is impossible, hence $0 \in N$.

Next, we assume that $n \in N$ and prove that $n^+ \in N$. To do so, we first assume that contrary—that is, we suppose that $(n, x) \in \gamma$, $(n^+, f(x)) \in \gamma$, and $(n^+, u) \in \gamma$ where $u \neq f(x)$. Let $\gamma^\circ = \gamma - \{(n^+, u)\}$; γ° satisfies (3) because $(n^+, u) \neq (0, c)$ (indeed, $n^+ \neq 0$ by Theorem 6.2). To show that γ° satisfies (4), suppose $(m, v) \in \gamma^\circ$; then $(m, v) \in \gamma$, so $(m^+, f(v)) \in \gamma$. Now we consider two cases, according as (a) $m^+ \neq n^+$ or (b) $m^+ = n^+$.

a) $m^+ \neq n^+$. Then $(m^+, f(v)) =(n^+, u)$, so $(m^+, f(v)) \in \gamma^\circ$.

b) $m^+ = n^+$. Then $m = n$ by 6.7, so $(m, v) = (n, v)$; but $n \in N$, so $(n, x) \in \gamma$ for no more than one $x \in A$; it follows that $v = x$, and so

$$(m^+, f(v)) = (n^+, f(x)) \in \gamma^\circ.$$

Thus, in either case (a) or (b), $(m^+, f(v)) \in \gamma^\circ$; thus γ° satisfies Condition (4), so $\gamma^\circ \in \mathscr{A}$. But γ is the intersection of all the elements of $\mathscr{A}$, so $\gamma \subseteq \gamma^\circ$; this is impossible, so we conclude that $n^+ \in N$. Thus $N = \omega$.

Finally, we will prove that γ is *unique*. Let γ and γ' be functions, from ω to A which satisfy I and II. We will prove by induction that $\gamma = \gamma'$. Let

$$M = \{n \in \omega : \gamma(n) = \gamma(n)\}.$$

Now $\gamma(0) = c = \gamma(0)$, so $0 \in M$; next, suppose that $n \in M$. Then

$$\gamma(n^+) = f(\gamma(n)) = f(\gamma'(n)) = \gamma'(n^+),$$

hence $n^+ \in M$. ■

6.9 Corollary Let f, c, and γ be as in Theorem 6.8. If f is injective and $c \notin \operatorname{ran} f$, then γ is injective.

Proof. We wish to show that if $\gamma(m) = \gamma(n)$, then $m = n$; the proof is by induction on m.

i) $m = 0$. If $n = 0$ we are done; if $n \neq 0$, then $n = k^+$ for some $k \in \omega$, so

$$c = \gamma(0) = \gamma(m) = \gamma(n) = \gamma(k^+) = f(\gamma(k)),$$

which is impossible because c is not in the range of f. Thus $n = 0 = m$.

ii) Suppose the corollary is true for m; let $\gamma(m^+) = \gamma(n)$. If $n = 0$ then we have $\gamma(0) = \gamma(m^+)$, which, as we have just shown, is impossible; thus $n \neq 0$, so $n = k^+$ for some $k \in \omega$. Thus $\gamma(m^+) = \gamma(k^+)$, that is, $f(\gamma(m)) = f(\gamma(k))$; but f is injective, so $\gamma(m) = \gamma(k)$. By the hypothesis of induction, it follows that $m = k$; hence $m^+ = k^+ = n$. ■

EXERCISES 6.3

1. Let $a \in \mathbb{R}$ where $\mathbb{R}$ is the set of the real numbers; define a^n by the following two conditions.

$$a^0 = 1,$$
$$a^{n+1} = a^n a, \quad \forall n \in \omega.$$

Prove that for each $n \in \omega$, a^n is a uniquely defined real number. (In other words, prove that $\gamma(n) = a^n$ is a uniquely determined function $\omega \to \mathbb{R}$; use Theorem 6.8.)

2. Let A be a set and let $f : A \to A$ be a function. Define f^n by

$$\begin{aligned} f^0 &= I_A && \text{(the identity function on } A\text{)}, \\ f^{n+1} &= f^n \circ f, && \forall n \in \omega. \end{aligned}$$

Prove that for each $n \in \omega$, f^n is a uniquely determined element of A^A .

3. Let A be a set and let $f : A \to B$ be an injective function, where $B \subset A$. Prove that A has a subset which is in one-to-one correspondence with ω. [*Hint*: Use 6.9 to prove that there is an injective $\gamma : \omega \to D$ where $D \subseteq A$.]

4. If A is partially ordered set, by a *strictly increasing sequence* in A we mean a function $\gamma : \omega \to A$ such that $\gamma(0) < \gamma(1) < \gamma(2) < \cdot\cdot\cdot$ Let A be a partially ordered set which has no maximal elements; prove that there is a strictly increasing sequence in A.

4 ARITHMETIC OF NATURAL NUMBERS

One of the most important applications of the recursion theorem is its use in defining addition and multiplication of natural numbers.

If m is a natural number, the recursion theorem guarantees the existence of a unique function $\gamma_m : \omega \to \omega$ defined by the two Conditions

I. $\gamma_m(0) = m$,

II. $\gamma_m(n^+) = [\gamma_m(n)]^+$, $\forall\, n \in \omega$.

Addition of natural numbers is now defined as follows:

$$m + n = \gamma_m(n)$$

for all $m, n \in \omega$. Conditions I and II immediately above can be rewritten thus:

6.10

$$\begin{aligned} m + 0 &= m, \\ m + n^+ &= (m + n)^+. \end{aligned}$$

We proceed to derive a few simple properties of addition.

6.11 Lemma $n^+ = 1 + n$, where 1 is defied to be 0^+.

Proof. This can be proven by induction on n. If $n = 0$, then we have

$$0^+ = 1 = 1 + 0$$

(this last equality follows from 6.10), hence the lemma holds for $n = 0$. Now, assuming the lemma is true for n, let us show that it holds for n^+:

$$\begin{aligned} 1+n^+ &= (1+n)^+ && \text{by 6.10} \\ &= (n^+)^+ && \text{by the hypothesis of induction. } \blacksquare \end{aligned}$$

6.12 Lemma $0 + n = n$.

Proof. Let $X = \{n \in \omega : 0 + n = n\}$; it will be shown by induction that $X = \omega$. Indeed, $0 + 0 = 0$ by 6.10, hence $0 \in X$. Now suppose that $n \in X$, that is, $0 + n = n$. Then

$$\begin{aligned} 0+n^+ &= (0+n)^+ && \text{by 6.10} \\ &= n^+ && \text{by the hypothesis of induction.} \end{aligned}$$

It follows by Theorem 6.3 that $X = \omega$. ■

6.13 Theorem $(m + n)+ k = m + (n + k)$.

Proof. The proof is by induction. For arbitrary elements $m, n \in \omega$, let

$$L_{mn} = \{k \in \omega : (m+n)+k = m+(n+k)\};$$

it will be shown that $L_{mn} = \omega$. First,

$$(m+n)+0 = m+n = m+(n+0);$$

hence $0 \in L_{mn}$. Now suppose $k \in L_{mn}$, that is,

$$(m+n)+k = m+(n+k).$$

Then

$$\begin{aligned} (m+n)+k^+ &= ((m+n)+k)^+ && \text{by 6.10} \\ &= (m+(n+k))^+ && \text{because } k \in L_{mn} \\ &= m+(n+k)^+ && \text{by 6.10} \\ &= m+(n+k^+) && \text{by 6.10,} \end{aligned}$$

so $k^+ \in L_{mn}$. ■

6.14 Theorem $m + n = n + m$.

Proof. For an arbitrary natural number m, let

$$L_m = \{n \in \omega : m+n = n+m\}.$$

It will be proven by induction that $L_m = \omega$. Now $m + 0 = m = 0 + m$ by 6.10 and 6.12, hence $0 \in L_m$.

Next suppose $n \in L_m$, that is, $m + n = n + m$. Then

$$\begin{aligned} m+n^+ &= (m+n)^+ && \text{by 6.10} \\ &= (n+m)^+ && \text{by the hypothesis of induction} \\ &= 1+(n+m) && \text{by 6.11} \\ &= (1+n)+m && \text{by 6.13} \\ &= n^+ + m && \text{by 6.11. } \blacksquare \end{aligned}$$

If m is a natural number, the recursion theorem guarantees the existence of a unique function $\beta_m : \omega \to \omega$ defined by the two Conditions

I. $\beta_m(0) = 0$,

II. $\beta_m(n^+) = \beta_m(n) + m, \forall n \in \omega$.

Multiplication of natural numbers is now defined as follows:

$$mn = \beta_m(n)$$

for all $m, n \in \omega$. Conditions I and II immediately above can be rewritten thus:

6.15

$$\begin{aligned} m0 &= 0, \\ mn^+ &= mn + m. \end{aligned}$$

The following are a few simple properties of multiplication.

6.16 Lemma $0n = 0$.

Proof. Let $N = \{n \in \omega : 0n = 0\}$; it will be shown by induction that $N = \omega$. Indeed, $00 = 0$ by 6.15, hence $0 \in N$. Now suppose that $n \in N$, that is, $0n = 0$. Then by 6.15, 6.10, and the hypothesis of induction,

$$0n^+ = 0n + 0 = 0n = 0;$$

hence $n^+ \in N$. It follows by induction that $N = \omega$. ■

6.17 Lemma $1n = n$.

The proof (by induction) is left as an exercise for the reader.

6.18 Theorem (*Distributive Law*).

i) $m(n + k) = mn + mk$.

ii) $(n + k)m = nm + km$.

Proof

i) If m and n are natural numbers, let

$$L_{mn} = \{k \in \omega : m(n+k) = mn + mk\}.$$

It will be shown by induction that $L_{mn} = \omega$. Now

$$m(n+0) = mn = mn + 0 = mn + m0,$$

hence $0 \in L_{mn}$. Next, suppose that $k \in L_{mn}$, that is, $m(n + k) = mn + mk$; then, by 6.10, 6.13 and 6.15,

$$\begin{aligned} m(n+k^+) &= m(n+k)^+ = m(n+k) + m \\ &= (mn+mk) + m = mn + (mk+m) = mn + mk^+, \end{aligned}$$

hence $k^+ \in L_{mn}$.

ii) The proof is left as an exercise for the reader.

6.19 Theorem (*Associative Law for Multiplication*). $(mn)\mathrm{k} = m(nk)$.
Proof. Let $L_{mn} =\{k \in \omega : (mn)k = m(nk)\}$; it will be proved by induction that $L_{mn} = \omega$. First, by 6.15,

$$(mn)0 = 0 = m0 = m(n0),$$

hence $0 \in L_{mn}$. Next, assume that $k \in L_{mn}$, that is $(mn)k= m(nk)$. Then, by 6.15 and 6.18,

$$\begin{aligned} (mn)k^+ &= (mn)k + mn = m(nk) + mn \\ &= m(nk+n) = m(nk^+); \end{aligned}$$

hence $k^+ \in L_{mn}$.■

6.20 Theorem (*Commutative Law for Multiplication*). $mn = nm$.
The proof is left as an exercise for the reader.

One of the most important aspects of the natural numbers is their ordering. Before proceeding with a formal definition of the order relation in ω, the reader should review the definition of ω. Specifically, it should be noted that for each n, the natural number n is the set of all the natural numbers preceding n:

$$n = \{0, 1, \ldots, n-1\}.$$

It is clear, now, how we are to define order in ω : n is to precede m if and only if n is an element of m. Motivated by this observation, we make the following formal definition.

6.21 Definition A relation $\leqslant$ is defined in ω as follows:

$$m \leqslant n \text{ if and only if } m \in n \text{ or } m = n.$$

First, it is required to prove that this is indeed an order relation in ω.

6.22 Theorem Let $m \leqslant n$ denote the fact that $m \in n$ or $m = n$. Then the relation $\leqslant$ is an order relation in ω.

Proof

i) For each $m \in \omega$, $m = m$, hence $m \leqslant m$ (reflexive law).

ii) Suppose $m \leqslant n$ and $n \leqslant m$; this means that either $m = n$, or $m \in n$ and $n \in m$. In the latter case, $m \subseteq n$ and $n \subseteq m$ by 6.6, hence again $m = n$ (antisymmetric law).

iii) Suppose $m \leqslant n$ and $n \leqslant p$; we have four possible cases.

1. $m \in n$ and $n \in p$: thus $m \in n$ and $n \subseteq p$, so $m \in p$.
2. $m \in n$ and $n = p$: thus $m \in p$.
3. $m = n$ and $n \in p$: thus $m \in p$.
4. $m = n$ and $n = p$: thus $m = p$.

In each case, $m \leqslant p$ (transitive law). ■

We will show next that ω is well ordered; this will require the following two lemmas.

6.23 Lemma If m is a natural number, $0 \leqslant m$.

Proof. Let $L = \{m \in \omega : 0 \leqslant m\}$; by the reflexive law 6.22(i), $0 \leqslant 0$, so $0 \in L$. Now suppose $m \in L$, that is, $0 \leqslant m$. From $m \in m^+$ it follows that $m \leqslant m^+$; thus by the transitive law 6.22(iii), $0 \leqslant m^+$, so $m^+ \in L$. So by 6.3, $L = \omega$. ■

6.24 Lemma If $n < m$ then $n^+ \leqslant m$.

Proof. If n is a natural number, let

$$L_n = \{m \in \omega : n < m \Rightarrow n^+ \leqslant m\};$$

We will use induction (6.3) to prove that $L_n = \omega$. Note that $m \notin L_n$ iff $n < m$ and $n^+ \nleqslant m$, that is, $n \in m$ and $n^+ \nleqslant m$. In particular, $0 \notin L_n$ iff $n \in 0$ and n^+ 0, which is impossible (specifically, $n \in 0$ is impossible); thus $0 \in L_n$. Now assume that $m \in L_n$, that is, $nm \Rightarrow n^+ \leqslant m$, and let us show that $m^+ \in L_n$, that is,

$$n < m^+ \Rightarrow n^+ \leqslant m^+.$$

If $n < m^+$, that is, $n \in m^+$, then by 6.4, $n \in m$ or $n = m$. If $n = m$, then $n^+ = m^+$, and we are done. If $n \in m$, that is, $n < m$, then by the hypothesis of induction, $n^+ \leqslant m < m^+$, so we are done again. ■

6.25 Theorem ω is well ordered.

Proof. Suppose, on the contrary, that A is a nonempty subset of ω without a least element. Let

$$L_n = \{n \in \omega : n \leqslant m \text{ for every } m \in A\}.$$

By 6.23, $0 \in L$. Now suppose that $n \in L$, that is, $n \leqslant m$ for every $m \in A$. If $n = p$ for some $p \in A$, then p is the least element of A, contrary to our hypothesis; thus nm for every $m \in A$. It follows by 6.24 that $n^+ \leqslant m$ for every $m \in A$, so $n^+ \in L$. Thus by 6.3, $L = \omega$. But $L \cap A = \varnothing$ because A has no least element. Thus $A = \varnothing$. ■

EXERCISES 6.4

1. Prove Theorem 6.18(ii).
2. Prove Theorem 6.20.
3. Prove each of the following.
   a) $m = n \Rightarrow m + k = n + k$ (see Exercise 3, Exercise Set 6.2),
   b) $m = n \Rightarrow mk = nk$.
4. Prove each of the following.
   a) $m < 1 \Rightarrow m = 0$.
   b) There is no natural number k such that $m < k < m^+$.
5. Prove each of the following.
   a) $n < k \Rightarrow m + n < m + k$.
   b) $m + n = m + k \Rightarrow n = k$.
6. Prove each of the following.
   a) If $m < n$ and $k \neq 0$, then $mk < nk$.
   b) If $mk = nk$ and $k \neq 0$, then $m = n$.
7. Prove that if $m \leqslant n$, then there exists a unique $p \in \omega$ such that $m + p = n$.
8. Prove each of the following.
   a) $m + k < n + k \Rightarrow m < n$.
   b) $mk < nk \Rightarrow m < n$.
9. Give an inductive definition of exponentiation of natural numbers; that is, define m^n in a manner similar to 6.10 and 6.15, justifying your definition in terms of the recursion theorem. Then prove each of the following.
   a) $m^{n+k} = m^n m^k$, b) $(mn)^k = m^k n^k$, c) $(m^n)^k = m^{nk}$.

5 CONCLUDING REMARKS

By Axiom A11, there exists a successor *set* X; by 6.1, $\omega \subseteq X$, hence by Axiom A3, *ω is a set*. It follows, by Axiom A3 again, that *every natural number is a set*.

In the next chapter we will define a class to be *finite* if it is in one-to-one correspondence with a natural number; it follows by 2.36 that *every finite class is a set*. In view of this remark, we will henceforth speak of finite *sets* rather than finite classes.

We have just seen that ω is a set; thus, by 1.53, $\omega \times \omega$ is a set. Now, if we identify each fraction n/m (where n/m is assumed to be in "lowest terms") with the ordered pair (n, m), then the class of all the positive rational numbers is a subclass of $\omega \times \omega$, hence by A3, it is a set. Analogously, the class of all the negative rational numbers is a set, hence by A6, the class $\mathbb{Q}$ of all the rational numbers is a set. It is well known that every real number can be regarded as a sequence (called a *Cauchy sequence*) of rational numbers; hence, roughly speaking,[*] as an element of $\mathscr{P}(\mathbb{Q})$. In other words, the class $\mathbb{R}$ of the real numbers is a subclass of $\mathscr{P}(\mathbb{Q})$; hence, by A7 and A3, it is a set. In similar fashion, the class $\mathbb{C}$ of the complex numbers is a set.

We have seen in preceding sections that the union of any set of sets is a set; if A is a set, then $\mathscr{P}(A)$is a set; if $\{A_i\}_{i\in I}$ is a family of sets, where I is a set, then $\prod_{i\in I} A_i$is a set; and if A and B are sets, then A^B is a set. Thus it is clear that every object we can produce by the classical construction processes is a set.

All the infinite (that is, not finite) classes which occur in traditional mathematics are sets; thus, in the next chapter, we will confine our attention to finite and infinite *sets*.

[*] To be more precise, every Cauchy sequence is an element of Q^{ω}.

7

Finite and Infinite Sets

1 INTRODUCTION

One of the most fundamental distinctions in mathematics is that between finite and infinite sets. The distinction is so intuitively compelling that, even in the absence of a precise definition, there cannot be any doubt as to whether a given set is finite or infinite. In simple terms, a finite set is one which "has n elements," where n is a natural number, and an infinite set is one which is not finite.

Although the dichotomy between finite and infinite has always fascinated mathematicians—it has been the source of the most celebrated riddles, paradoxes, and classical errors of mathematics—a sound theory of infinite sets did not appear until very recent times. It had to await the arrival of the rigorous concepts of set mapping, and one-to-one correspondence. Once the use of these new tools became familiar to mathematicians, toward the end of the nineteenth century, the modern theory of infinite sets developed rapidly; it was largely the work of Georg Cantor and his successors.

Using familiar concepts, and arguments which are remarkable for their simplicity, Cantor was able to draw conclusions which surprised mathematicians and laymen alike. Cantor's ideas are well known today; they have been popularized in innumerable expository books and articles, and have entered the lore of modern mathematics. We proceed, in the remainder of this section, to give the bare outlines of Cantor's theory. In this discussion the words "finite" and "infinite" will be used informally; as we remarked earlier, a finite set can be described as one which "has n elements" (n is a natural number), and an infinite set is, simply, one which is not finite.

Two finite sets A and B have the "same number of elements" if and only if they are in one-to-one correspondence. Even though we cannot speak of two infinite sets as having the "same number of elements," we have the feeling, nonetheless, that if A and B are infinite sets and there is a one-to-one correspondence between them, then, in a certain sense, they are of the "same size." This intuitive notion is formalized by defining two sets A and B to be *equipotent*, or to have the *same power*, if there is a one-to-one correspondence from A to B. We say that A is of a *lesser power* than B if there exists a one-to-one correspondence between A and a proper subset of B, but none between A and B. Here, again, we have taken our cue from the finite case: for if A and B are finite sets and A has fewer elements than B, then certainly there exists a one-to-one correspondence between A and a part of B, but none between A and all of B. If A and B have the same power, we write $A \approx B$; if the power of A is less than that of B, we write $A \prec B$; finally, if the power of A is less than or equal to that of B (that is, A is in one-to-one correspondence with a subset of B), then we write $A \preccurlyeq B$.

It is a curious fact, first proven by Cantor, that many sets which appear to be smaller—or larger—than ω actually have the same power as ω. For example, if E is the set of the even natural numbers, it is easy to see that the function $f(n) = 2n$ is a bijective function from ω to E. Thus, although E is a proper subset of ω (in fact, E appears to have only "half as many" elements as ω), actually E is equipotent with ω. A more surprising example involves ω and the set $\mathbb{Q}$ of the rational numbers; our intuition suggests, in the most compelling way, that $\mathbb{Q}$ is a "larger" set than ω; for $\mathbb{Q}$ not only includes ω, but is, in an obvious manner, "infinitely dense" with respect to ω. Yet it can easily be shown that ω and $\mathbb{Q}$ are in one-to-one correspondence; the proof consists in "enumerating" the rational numbers—that is, making a list $r_1, r_2, r_3, \ldots$ of rational numbers which includes them all; the correspondence $i \leftrightarrow r_i$ is then a one-to-one correspondence between ω and $\mathbb{Q}$. We proceed as follows.

First, we group all the positive rational numbers into classes A_1, A_2, … , where A_i contains all the fractions n/m such that $n + m = i$. Within each class A_i, we order the numbers n/m in increasing order of the numerator n. Thus the first few fractions in this ordering would be as follows:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{1}{5}, \ldots .$$

Now we delete all fractions which are not in “lowest terms;” this leaves

$$1, \tfrac{1}{2}, 2, \tfrac{1}{3}, 3, \tfrac{1}{4}, \tfrac{2}{3}, \tfrac{3}{2}, \ldots .$$

This is clearly an enumeration r_1, r_2, … of the positive rational numbers. If t_1, t_2, … is a similar enumeration of the negative rational numbers, then 0, r_1, t_1, r_2, t_2, … is an enumeration of all the rational numbers.

A set which is in one-to-one correspondence with ω is said to be *denumerable*. Faced with the unexpected discovery that $\mathbb{Q}$ is denumerable, we are naturally led to wonder whether every infinite set is denumerable. This question was answered by Cantor—in the negative: the set of all the real numbers, for example, is not denumerable. To prove this, we use the so-called *diagonal method*.

First, we note that the function $y = \tan(\pi x - \pi/2)$ is a one-to-one correspondence between the set of all the real numbers and the open interval (0, 1); hence it will be sufficient for our purposes to prove that the set of all the real numbers r such that $0 < r < 1$ cannot be enumerated. We argue by contradiction. Suppose that r_1, r_2, … is an enumeration of all the real numbers between 0 and 1. Let each real number be expressed as a nonterminating decimal; thus $r_i = \cdot r_{i1} r_{i2} r_{i3} \cdots$, where each r_{ij} is a digit 0, 1, … , 9.

$$\begin{aligned} r_1 &= \cdot r_{11} r_{12} r_{13} r_{14} \cdots \\ r_2 &= \cdot r_{21} r_{22} r_{23} r_{24} \cdots \\ r_3 &= \cdot r_{31} r_{32} r_{33} r_{34} \cdots \\ r_4 &= \cdot r_{41} r_{42} r_{43} r_{44} \cdots \\ &\vdots \end{aligned}$$

Now we define s to be a number $s = \cdot s_1 s_2 s_3 \ldots$, where $s_1 \neq r_{11}$, $s_2 \neq r_{22}$, $s_3 \neq r_{33}$, $s_4 \neq r_{44}$, and so forth (for example, we might dictate that $s_i = 1$ if $r_{ii} \neq 1$ and $s_i = 2$ if $r_{ii} \neq 1$). Now, since $0 < s < 1$, it follows that s is one of the numbers in the enumeration, say $s = r_k$. But this is impossible, because the kth digit of s is s_k and the kth digit of r_k is r_{kk}, and $s_k \neq r_{kk}$. Because of this contradiction, it is clear that there is no way of enumerating the real numbers. Yet ω has the same power as a subset of the real numbers—namely those real numbers which happen to be positive integers. Thus $\omega \prec \mathbb{R}$.

We have just revealed one of the most significant facts in Cantor’s theory of the infinite: while ω and $\mathbb{R}$ are both infinite sets, one of them is strictly larger than the other; in other words infinite sets, like finite sets, come in different “sizes.” Our next step, naturally, is to find a set which is strictly larger than $\mathbb{R}$. In order to settle this question, however, we introduce a result of far greater generality, which will provide us at once with a strictly increasing sequence $K_1 \prec K_2 \prec K_3 \prec \ldots$ of infinite sets. Our result is simply this: if A is any set, then the power set of A, $\mathscr{P}(A)$, is strictly larger than A. To prove this, we use a variant of the diagonal method which served us in the preceding paragraph.

We begin by assuming that there is a one-to-one correspondence $\phi : A \to \mathscr{P}(A)$. We define a set B as follows:

$$B = \{x \in A : x \notin \phi(x)\};$$

B is a subset of A, so $B = \phi(y)$ for some $y \in A$. Now if $y \in \phi(y)$, then $y \notin B$, that is, $y \notin \phi(y)$; yet if $y \notin \phi(y)$, then $y \in B$, that is, $y \in \phi(y)$. Quite obviously this is impossible; hence there exists no one-to-one correspondence between A and $\mathscr{P}(A)$. However, A has the same power as a subset of $\mathscr{P}(A)$, namely the set of all the singletons $\{x\}$. We conclude that $A \prec \mathscr{P}(A)$.

The argument of the preceding paragraph is a proof for the following theorem.

7.1 Theorem If A is a set, there exists no surjective function $A \to \mathscr{P}(A)$.

7.2 Corollary No subset of A can be equipotent with $\mathscr{P}(A)$.

7.3 Corollary A cannot be equipotent with any set containing $\mathscr{P}(A)$.

It follows from 7.2 and 7.3 that

7.4 if $B \subseteq A$, then $B \prec \mathscr{P}(A)$;

if $\mathscr{P}(A) \subseteq D$, then $A \prec D$.

We have proved earlier (Theorem 2.35) that if A is a set, $\mathscr{P}(A)$ and 2^A are in one-to-one correspondence. Thus all of the above statements hold true when we replace $\mathscr{P}(A)$by 2^A.

If we let $\omega = K_1$, $P(K_1) = K_2$, $P(K_2) = K_3$, and so forth, then we have the strictly increasing sequence of infinite sets

$$K_1 \prec K_2 \prec K_3 \prec \cdots$$

Now consider $L_1 = \bigcup_{i\in\omega} K_i$: for each i, $K_{i+1} \subseteq L_1$, that is, $\mathscr{P}(K_i) = L_1$. It follows by 7.4 that for each i, $K_i \prec L_1$. Now we let $L_2 = \mathscr{P}(L_1)$, $L_3 = \mathscr{P}(L_2)$, and so forth; hence we have the strictly increasing sequence of infinite sets

$$K_1 \prec K_2 \prec K_3 \prec \cdots \prec L_1 \prec L_2 \prec L_3 \prec \cdots \prec M_1 \prec M_2 \prec M_3 \prec \cdots$$

Thus, speaking informally, there are many more "sizes" of infinite sets than there are different "sizes" of finite sets.

It is worth nothing that the set $\mathbb{R}$ of the real numbers is equipotent with 2^ω. Indeed, we have already noted that $\mathbb{R}$ is equipotent with the open interval (0, 1) of $\mathbb{R}$, hence with the closed interval [0, 1]. (This last fact follows trivially from Exercise 2 in Exercises 7.3.) Now, each element r in the interval [0, 1] can be written in binary notation

$$r = r_1 r_2 r_3 \ldots,$$

where each r_i, is either 0 or 1. This expression for r can be identified with the function $\phi_r : \omega \to \{0, 1\}$ given by $\phi_r(i) = r_i$, $\forall\ i \in \omega$. It is easy to see that the correspondence $r \leftrightarrow \phi_r$ is a one-to-one correspondence between the interval [0, 1] and 2^ω.

We have seen that the set $\mathbb{Q}$ of the rational numbers is equipotent with ω; it is easy to show that the set $\mathbb{C}$ of the complex numbers is equipotent with $\mathbb{R}$; thus, by the preceding paragraph, all of classical mathematics deals with only two sizes of infinite sets, namely, sets equipotent with ω and sets equipotent with 2^ω; the power of 2^ω is often called the *power of the continuum*. Now an interesting question which arises is the following: is there a power between that of ω and that of 2^ω? That is, does there exist any set A such that $\omega \prec A \prec 2^\omega$? Since no such set occurs anywhere in classical mathematics, and there appears to be no way of constructing one, it was conjectured by Cantor and his contemporaries that the answer to that question must be "no;" this conjecture is known as the *continuum hypothesis*. A closely related conjecture is the *generalized continuum hypothesis*, which proposes that for every set B, there is no set A such that $B \prec A \prec 2^B$. These hypotheses have never been either proven or disproven; we shall have more to say about them in Chapter 11.

The aim of this chapter is to exploit the various ideas which have been motivated in our introduction. We will define rigorously the concepts of finite and infinite set, "power," and cardinality, and give the classical results of Cantor's theory.

2 EQUIPOTENCE OF SETS

In the preceding section, we have defined the symbols $\approx$, $\prec$, and $\preccurlyeq$ as follows:

$A \approx B$ iff A is in one-to-one correspondence with B.

$A \preccurlyeq B$ iff A is in one-to-one correspondence with a subset of B.

$A \prec B$ iff A is in one-to-one correspondence with a subset of B and A is not in one-to-one correspondence with B.

It is immediate from the second of these statements that

7.5 $A \preccurlyeq B$ iff there exists an injective function $A \to B$.

Furthermore, we have the following.

7.6 Lemma There exists an injective function $f : A \to B$ if and only if there exists a surjective function $g : B \to A$.

Proof

i) Suppose $f : A \to B$ is injective; by 2.25, there exists a function $g : B \to A$ such that $g \circ f = I_A$; thus, by 5.4, g is surjective.

ii) Suppose $g : B \to A$ is surjective; by 5.4, there exists a function $f : A \to B$ such that $g \circ f = I_A$; thus, by 2.25, f is injective. ■

By 7.5 and 7.6 we have

7.7 $A \preccurlyeq B$ iff there exists a surjective function $B \to A$.

7.8 Theorem Let A, B, C, and D be sets where $A \cap C = \varnothing$ and $B \cap D = \varnothing$. If $f : A \to B$ and $g : C \to D$ are bijective functions, then $f \cup g$ is a bijective function $A \cup C \to B \cup D$.

Proof. If $f : A \to B$ and $g : C \to D$ are functions, then clearly $f : A \to B \cup D$ and $g : C \to B \cup D$ are functions, hence by 2.16,

$$(f \cup g) : A \cup C \to B \cup D$$

is a function. Now $f : A \to B$ and $g : C \to D$ are bijective, hence by 2.21,

$$f^{-1} : B \to A \quad \text{and} \quad g^{-1} : D \to C$$

are functions; thus, as above,

$$(f^{-1} \cup g^{-1}) : B \cup D \to A \cup C$$

is a function. But clearly $f^{-1} \cup g^{-1} = (f \cup g)^{-1}$, hence

$$(f \cup g)^{-1} : B \cup D \to A \cup C$$

is a function; thus, by 2.22,

$$(f \cup g) : A \cup C \to B \cup D$$

is bijective. ■

7.9 Corollary Suppose $A \cap C = \varnothing$ and $B \cap D = \varnothing$; if $A \approx B$ and $C \approx D$, then $A \cup C \approx B \cup D$.

7.10 Theorem If $A \approx B$ and $C \approx D$, then $A \times C \approx B \times D$.

Proof. Let $f : A \to B$ and $g : C \to D$ be bijective functions, and let us define $h : A \times C \to B \times D$ as follows:

$$h(x, y) = (f(x), g(y)), \forall (x, y) \in A \times C.$$

It can easily be shown that $h : A \times C \to B \times D$ is bijective; the details are left to the reader. ■

7.11 Theorem If $A \approx B$ and $C \approx D$, then $A^C \approx B^D$.

Proof. Let $f : A \to B$ and $g : D \to C$ be bijective functions, and let us define $h : A^C \to B^D$ in the following way. For each $\alpha \in A^C$, that is, for each function $\alpha : C \to A$, let $h(\alpha) = f \circ \alpha \circ g$; clearly $f \circ \alpha \circ g$ is a function $D \to B$, that is, $f \circ \alpha \circ g \in B^D$. It can be shown routinely that $h : A^C \to B^D$ is a bijective function; the details are left to the reader. ■

7.12 Corollary If $A \approx B$, then $\mathscr{P}(A) \approx \mathscr{P}(B)$.

This follows immediately from 7.11 and 2.35.

EXERCISES 7.2

1. Complete the proof of Theorem 7.10.
2. Complete the proof of Theorem 7.11.
3. Prove that if $(A - B) \approx (B - A)$, then $A \approx B$.
4. Suppose $A \approx B$, $a \in A$, and $b \in B$. Prove that $(A - \{a\}) \approx (B - \{b\})$.
5. Suppose that $A \approx B$, $C \approx D$, $C \subset A$ and $D \subset B$. Prove that $(A - C) \approx (B - D)$.
6. Let $\{B_i\}_{i\in I}$ and $\{C_i\}_{i\in I}$ each be a family of mutually disjoint sets. If $B_i \approx C_i$ for each $i \in I$, prove that

$$\bigcup_{i\in I} B_i \approx \bigcup_{i\in I} C_i,$$

7. Let $\{B_i\}_{i\in I}$ and $\{C_i\}_{i\in I}$ be families of sets. If $B_i \approx C_i$ for each $i \in I$, prove that

$$\prod_{i\in I} B_i \approx \prod_{i\in I} C_i.$$

3 PROPERTIES OF INFINITE SETS

A set A is said to be *finite* if A is in one-to-one correspondence with a natural number n; otherwise, A is said to be *infinite*. Several other definitions of “finite” and “infinite” are to be found in the mathematical literature; foremost among them are the following:

i) A is infinite if and only if A has a denumerable subset.

ii) A is infinite if and only if A is equipotent with a proper subset of itself.

In each of the above two cases, a set is called “finite” if it is not infinite.

It will be shown next that (i) and (ii) are each equivalent to our definition of “infinite,” given above.

7.13 Lemma If A is a denumerable set and $x \in A$, then $A - \{x\}$ is a denumerable set.

Proof. If A is denumerable, then there exists a bijective function $f : \omega \to A$. Corresponding to x, there is an $n \in \omega$ such that $f(n) = x$; define $g : \omega \to A$ as follows:

$$g(m) = \begin{cases} f(m) & \text{if} \quad m < n, \\ f(m+1) & \text{if} \quad m \geqslant n. \end{cases}$$

It is easy to see that g is a bijective function from ω to $A - \{x\}$; the details are left to the reader. ■

7.14 Theorem *A* is an infinite set if and only if *A* has a denumerable subset.

Proof

i) Well-order *A*; by Theorem 4.62, exactly one of the following cases holds: (α) ω is isomorphic with *A*; (β) ω is isomorphic with an initial segment of *A*; (γ) *A* is isomorphic with an initial segment of ω. If *A* does not have a denumerable subset, then (α) and (β) cannot hold, hence (γ) holds; therefore *A* is equipotent with an initial segment $n = S_n$ of ω, so *A* is finite.* We have just proved that if *A* does *not* have a denumerable subset, then *A* is finite.

ii) To prove the converse, we will first use induction to show that a natural number *n* cannot have a denumerable subset. This assertion is clearly true for $n = 0$; let it be true for *n*, and suppose n^+ has a denumerable subset *S*. If $n \notin S$, then *S* is a denumerable subset of *n* (recall that "*n*" was defined to be the set $\{0, 1, \ldots, n-1\}$); by the hypothesis of induction, this cannot happen. If $n \in S$, then $S - \{n\} \subseteq n$; but $S - \{n\}$ is denumerable (Lemma 7.13), so by the hypothesis of induction this cannot happen. We conclude that n^+ cannot have a denumerable subset. Now suppose that *A* has a denumerable subset *B* and *A* is finite; that is, $A \approx n$, $B \subseteq A$, and $B \approx \omega$. Then we have injective functions as follows: $\omega \to B \to A \to n$; their composite is an injective function $\omega \to n$, and we have just proven this to be impossible. Thus if *A* has a denumerable subset, then *A* is infinite. ■

7.15 Corollary Every set which has an infinite subset is infinite.

7.16 Corollary Every subset of a finite set is finite.

7.17 Corollary If *A* is an infinite set and *B* is nonempty, then $A \times B$ and $B \times A$ are infinite sets.

Proof. If *y* is a fixed element of *B*, the function $f : A \to A \times B$ given by $f(x) = (x, y)$ is clearly injective. Thus if $g : \omega \to A$ is a bijective function, then $f \circ g : \omega \to A \times B$ is injective. It follows that $A \times B$ has a denumerable subset, hence $A \times B$ is infinite. ■

7.18 Theorem *A* is an infinite set if and only if *A* is equipotent with a proper subset of itself.

Proof

i) Suppose that *A* is infinite; by Theorem 7.14, *A* has a denumerable subset $B = \{a_0, a_1, a_2, \ldots\}$. Let the function $f : A \to A$ be defined by

$$f(x) = x, \quad \forall x \in A - B,$$
$$f(a_m) = a_{m+1}, \quad \forall m \in \omega.$$

Clearly, *f* is a one-to-one correspondence between *A* and $A - \{a_0\}$.

ii) Suppose there exists a bijective function $f : A \to B$, where B is a proper subset of A. Let c be an arbitrary element of $A - B$; by the recursion theorem, there exists a function $\gamma : \omega \to A$ which satisfies the conditions

$$(\alpha)\ \gamma(0) = c \text{ and } (\beta)\ \gamma(n^+) = f(\gamma(n)).$$

Now ran $f = B$ and $c \in A - B$, so $c \notin$ ran f; thus by 6.9, γ is injective. The range of γ is obviously a denumerable subset of A, so by 7.14, A is infinite. ■

EXERCISES 7.3

1. Let A and B be a pair of disjoint finite sets. Use induction to prove that if $A \approx m$ and $B \approx n$, then $A \cup B \approx m + n$. Conclude that the union of two finite sets is finite.
2. Using the result of Exercise 1, prove that if A is an infinite set and B is a finite subset of A, then $A - B$ is infinite. Prove $A - B \approx A$.
3. Prove that a natural number is not equipotent with a proper subset of itself. Conclude that if $A \approx m$ and $n > m$, then $A \not\approx n$.
4. Prove that A is an infinite set if and only if $\forall n \in \omega$, A has a subset B such that $B \approx n$.
5. Let A and B be finite sets. Use induction to prove that if $A \approx m$ and $B \approx n$, then $A \times B \approx mn$. Conclude that the Cartesian product of two finite sets is finite.
6. Assuming that A is an infinite set and B is denumerable, prove that $A \approx (A \cup B)$.
7. Suppose $x \in A$; prove that A is an infinite set if and only if $A \approx (A - \{x\})$.
8. Use induction to prove that if $A \approx n$, then $\mathscr{P}(A) \approx 2^n$. Conclude that if A is a finite set, then $\mathscr{P}(A)$ is a finite set.
9. Prove Corollary 7.15.
10. Prove Corollary 7.16.

4 PROPERTIES OF DENUMERABLE SETS

Once again, a set is called *denumerable* if it is in one-to-one correspondence with ω. The fundamental properties of denumerable sets are presented in this section.

7.19 Theorem Every subset of a denumerable set is finite or denumerable.

Proof. First we note that every subset of ω is finite or denumerable. Indeed, let $E \subseteq \omega$; by 4.63, either $E \simeq \omega$ or $E \simeq S_n = n$ for some $n \in \omega$. Now let A be a denumerable set and let $B \subseteq A$; there exists a bijective function $f : A \to \omega$. Now $f(B) \subseteq \omega$, so $f(B)$ is finite or denumerable; but f is bijective, so $B \approx f(B)$; hence B is finite or denumerable. ■

7.20 Theorem $\omega \times \omega \approx \omega$.

Proof. We will use the recursion theorem to establish the existence of a bijective function from ω to $\omega \times \omega$. Let $A = \omega \times \omega$; we define a function $f : A \to A$ as follows:

$$f(k,m) = \begin{cases} (0, k+1) & \text{if } m = 0, \\ (k+1, m-1) & \text{if } m \neq 0. \end{cases}$$

We note that *f* is injective, for suppose

$$f(k,m) = f(n,p) = (r,s).$$

If *r* = 0, then (because of the way *f* is defined) *m* = 0 and *p* = 0; hence

$$(0,s) = f(k,m) = (0,k+1) \quad \text{and} \quad (0,s) = f(n,p) = (0,n+1),$$

So *k* = *n*. If *r* ≠ 0, then

$$(r,s) = f(k,m) = (k+1, m-1) \quad \text{and} \quad (r,s) = f(n,p) = (n+1, p-1),$$

so *k* = *n* and *m* = *p*. Thus *f* is injective. Now we make use of the recursion theorem: we define a function $\gamma : \omega \to A$ by the two conditions

i) $\gamma(0) = (0, 0)$, and

ii) $\gamma(n^+) = f(\gamma(n))$.

We note (again, because of the way *f* is defined) that (0, 0) cannot be in the range of *f*; it follows by 6.9 that *γ* is injective. Finally, we show that *γ* is surjective; indeed, we will show that if $k, m \in \omega$, then $(k, m) = \gamma(n)$ for some $n \in \omega$. The proof is by induction on *k* + *m*:

I. If $k + m = 0$, then $(k, m) = (0, 0) = \gamma(0)$.

II. Suppose $k + m = n^+$; if *k* = 0, then $(k, m) = f(m - 1, 0)$; by the hypothesis of induction, $(m - 1, 0) = \gamma(q)$ for some $q \in \omega$, so

$$(k,m) = f(m-1,0) = f(\gamma(q)) = \gamma(q^+).$$

If k ≠ 0, then $(k, m) = f(k - 1, m + 1)$; by the hypothesis of induction, $(k + m - 1, 0) = \gamma(p)$ for some $p \in \omega$; thus

$$\gamma(p+1) = (0, k+m), \gamma(p+2) = (1, k+m-1), \ldots,$$
$$\gamma(p+k+1) = (k,m). \blacksquare$$

7.21 Corollary If *A* and *B* are denumerable sets, then *A* × *B* is a denumerable set.

Proof. If $A \approx \omega$ and $B \approx \omega$, then $A \times B \approx \omega \times \omega$ (7.10). But $\omega \times \omega \approx \omega$; thus $A \times B \approx \omega$. ■

7.22 Theorem Let $\{A_n\}_{n \in \omega}$ be a denumerable family of denumerable sets, and let

$$A = \bigcup_{n \in \omega} A_n;$$

then A is denumerable. [A denumerable union of denumerable sets is denumerable.]

Proof. To say that each A_n is denumerable means that there exists a family $\{f_n\}_{n\in\omega}$ of functions such that, $\forall n \in \omega$, $f_n : \omega \to A_n$ is bijective. We define $\sigma : \omega \times \omega \to A$ by: $\sigma(k, m) = f_k(m)$. It is easy to see that σ is surjective: for if $x \in A$, then $x \in A_n$ for some $n \in \omega$, and if $x \in A_n$, then $x = f_n(m) = \sigma(n, m)$ for some $m \in \omega$.

By Theorem 7.20, there exists a bijective function $\phi : \omega \to \omega \times \omega$; hence $\sigma \circ \phi : \omega \to A$ is surjective. It follows (7.7) that $A \approx E$ for some subset $E \subseteq \omega$; now E is either finite or denumerable (Theorem 7.19); hence A is either finite or denumerable. By Corollary 7.15, A is not finite, so A is denumerable. ■

7.23 Corollary The union of two denumerable sets is denumerable.

EXERCISES 7.4

1. Prove that the union of two denumerable sets is denumerable. (Corollary 7.23.)
2. Let A be a denumerable set. Prove that A has a denumerable subset B such that $A - B$ is denumerable.
3. Prove that $\omega^n \approx \omega$. [*Hint*: Use the definitions $\omega^1 = \omega$ and $\omega^{n+} = \omega^n \times \omega$; use 7.10, 7.20, and induction.] Conclude that if A is a denumerable set, then A^n is a denumerable set.
4. Prove that $\omega \cup \omega^2 \cup \omega^3 \cup \ldots$ is a denumerable set.
5. Prove that the set of all finite subsets of ω is denumerable. Then prove that the set of all finite subsets of a denumerable set is denumerable.
6. Let A be an infinite set. Prove that A is denumerable if and only if $A \approx B$ for every infinite subset $B \subseteq A$.
7. Prove that if A is a nonempty finite set and B is denumerable, then $A \times B$ is denumerable.
8. Let $\mathscr{L}$ be the set of all polynomials $a_0 + a_1x + \ldots + a_nx^n$ with integer coefficients. Prove that $\mathscr{L}$ is denumerable. [*Hint*: This may be proved by using an argument of the kind used in the introduction to prove that $\mathbb{Q}$ is denumerable.]
9. An *algebraic number* is any real root of an equation $a_0 + a_1x + \ldots + a_nx^n = 0$, where the coefficients a_i are integers. Prove that the set of all algebraic numbers is denumerable.
10. A real number is called *transcendental* if it is not algebraic. Prove that the set of all transcendental numbers is nondenumerable.
11. Use the results of Exercises 1 and 5, above, to prove that the set of all infinite subsets of ω is equipotent with 2^ω.

[*] Note that, by definition, the natural number n is the set $\{0, 1, 2, \ldots, n - 1\}$, that is, n is exactly the initial segment S_n of ω.

8

Arithmetic of Cardinal Numbers

1 INTRODUCTION

In the preceding chapter we defined what is meant by a finite set; it follows from our definition that *every finite set is equipotent with exactly one natural number n*. This fact has an important consequence, namely, that the natural numbers may be used as a set of standards—a scale, as it were—to measure the size of finite sets. If A is any finite set, then A may be "measured" by comparison with the natural numbers, and will be found to correspond—that is, to be equipotent—with exactly one of them. When the natural numbers serve in this capacity—as standards to measure the size of sets—they are commonly called *cardinal numbers*.

A natural and fascinating question arises now: Can we find a way of extending our system of cardinal numbers so as to create a set of standards for measuring the size of *all* sets? To put it another way: Can we define "infinite cardinal numbers," and can we construct a sufficient supply of them so that *every* set has a cardinal number (if A has n elements, we say that *A has cardinal number n*)? The answer is "yes:" We can generalize the concept of cardinal number with such remarkable ease that almost all of the properties of the finite cardinals—their ordering, their arithmetic, and so forth—apply as naturally to the infinite cardinals as they did to the finite ones. To take one example: What is meant by the sum $m + n$ of two cardinal numbers? The idea, clearly, is that if A has m elements and B has n elements—and if A and B are disjoint—then $m + n$ is the cardinal number of $A \cup B$. What could be more natural than to extend this notion of cardinal sum to all sets (or rather, to all "set sizes")?

Before giving a general definition of cardinal numbers, let us take one more look at the natural numbers and see why they can be used as standards to measure the size of finite sets. As we stated earlier, every finite set is equipotent with exactly one natural number n; that is, the natural numbers are well defined sets, and there is a unique natural number for each and every finite "set size." It is clear, now, what we expect of our definition of cardinal numbers: The cardinal numbers are to be *well-defined sets*, and *every set is to be equipotent with exactly one cardinal number*. It is immaterial what sets the cardinal numbers are; the only requirement is that there be exactly one cardinal number of each "size."

A simple way of constructing the cardinal numbers would be the following. We observe that the relation "A is equipotent with B" ($A \approx B$)is an equivalence relation among sets. Thus we might partition the class of all sets into equipotence classes, and select one representative of each class: the representatives would be our cardinal numbers. This process seems quite natural—and will, indeed, serve as the intuitive basis of our definition. However, it cannot be applied literally. Note, for example, that if A is a set, the equipotence class $\{B : B \approx A\}$ may be a proper class; hence it is not legitimate to speak of the "class of all the equipotence classes." Furthermore, even if we *could* speak of the "class of all the equipotence classes," it would not be legitimate to use the Axiom of Choice to pick a representative of each class; indeed, the Axiom of Choice (see statement Ch 1 on page 115) allows us to pick representatives from a *set of sets*, not from an arbitrary class of classes.

Since we cannot literally "select" our cardinals by using the Axiom of Choice, how are we to proceed? A simple way, and one which is sanctioned by mathematical tradition, is to *posit* their existence (that is, to posit the existence of a representative set from each "equipotence class") by means of a new axiom.

A12 Axiom of Cardinality There is a class CD of sets, called *cardinal numbers,* with the following properties:

K1 If A is any set, there exists a cardinal number a such that $A \approx a$.
K2 If A is a set and a, b are cardinal numbers, then $A \approx a$ and $A \approx b \Rightarrow a = b$.

We will add the Axiom of Cardinality to our list of axioms for set theory—but only on a provisional basis, for in the next chapter we will describe a method for *constructing* sets with properties K1 and K2—that is, we will produce actual sets (in much the same way as we produced the natural numbers) which will serve as cardinal numbers.

We will use lower-case Roman letters, such as a, b, c, d, etc., to denote cardinal numbers.

It is worth nothing, incidentally, that *the class CD of all the cardinal numbers is a proper class*. For suppose CD is a set: since each cardinal number is a set, it follows by Axiom A6 that

$$V = \bigcup_{a \in CD} a$$

that is, the union of all the cardinal numbers, is a set. Thus by Axiom A7, $\mathscr{P}(V)$ is a set; but then, by condition K1 of Axiom A12, there is a cardinal number e such that e $\approx \mathscr{P}(V)$. Now e $\in CD$, hence e $\subseteq V$, which is impossible by Corollary 7.2. This contradiction proves that CD is not a set, but a proper class.

2 OPERATIONS ON CARDINAL NUMBERS

If A is a set, a is a cardinal number and $A \approx a$, then we say that a is the cardinal number of A. We denote this by writing

$$a = \#A.$$

Now conditions K1 and K2 can be conveniently restated as follows:

K1 If A is any set, there exists a cardinal number a such that $a = \#A$.
K2 If A is a set and a, b are cardinal numbers, then $a = \#A$ and $b = \#A \Rightarrow a = b$.

8.1 Lemma If a and b are cardinal numbers and $a \approx b$, then $a = b$.

The proof is an immediate consequence of K2.

8.2 Lemma If $A \approx B$, then $\#A = \#B$.

Proof. By K1, there are cardinals a, b such that $a = \#A$ and $b = \#B$. Now $a \approx A$ and $b \approx B$; thus, if $A \approx B$, it follows that $a \approx A$ and $b \approx A$,so by K2, $a = b$.

We now proceed to define the addition and multiplication of cardinal numbers. Our definitions require no comment; they correspond in the most natural way to our intuitive understanding of the process of adding and multiplying whole numbers.

Let a and b be two cardinals. Let A and B be disjoint sets such that $a = \#A$ and $b = \#B$. Then $a + b$ is the cardinal number defined by

$$a+b=\#(A\cup B).$$

Note. In the preceding definition it has been assumed that we can always find *disjoint* sets A and B such that $a = \#A$ and $b = \#B$. This is obviously true. For example, take $A = a \times\{0\}$ and $B = b \times\{1\}$; then A consists of pairs $(x,0)$, whereas B consists of pairs $(x, 1)$.

Let a and b be two cardinals. Let A and B be sets such that $a = \#A$ and $b = \#B$. The ab is the cardinal number defined by

$$ab=\#(A\times B).$$

Note. Since a and b are sets, we can write $ab = \#(a \times b)$.

When introducing a new operation, it is necessary to show that the operation is well-defined. For cardinal addition and multiplication, this means the following.

For addition: If $A_1 \approx A$ and $B_1 \approx B$ then $A_1 \cup B_1 \approx A \cup B$.

For multiplication: If $A_1 \approx A$ and $B_1 \approx B$ then $A_1 \times B_1 \approx A \times B$.

This guarantees that the sum and product do not depend on the specific sets chosen, so long as their cardinality is the same. The proof of these two assertions are given as exercises at the end of the Section.

The usual algebraic laws for addition and multiplication follow from the elementary properties of sets.

8.3 Theorem If a, b, c are cardinal numbers, the following laws hold:

i) $a + b = b + a$,

ii) $ab = ba$,

iii) $a + (b+ c)= (a+ b)+ c$,

iv) $a(bc)= (ab)c$,

v) $a(b+ c)= ab + ac$.

Proof. Properties (i), (iii), and (v) are immediate consequences of Theorems 1.25(i), 1.25(v), and 1.32(ii) respectively. To prove (ii), it must be shown that there exists a one-to-one correspondence between $A \times B$ and $B \times A$; the function $\varphi(x, y)= (y, x)$is obviously such a correspondence. Finally, to prove (iv), we must show that there exists a one-to-one correspondence between $A \times (B\times C)$and $(A \times B) \times C$; the function

$$\phi(x,(y,z))=((x,y),z)$$

is clearly such a correspondence.

Let A and B be finite sets. In Chapter 2 we defined A^B to be the set of all functions from B to A. Now suppose that A has m elements and B has n elements, and consider the process of constructing an arbitrary function from B to A. Since B has n elements, and each element can be assigned an image in m

possible ways, this means there are exactly m^n distinct functions from B to A. This simple observation suggests the following definition of cardinal exponentiation:

Let a and b be two cardinals. Let A and B be sets such that $a = \#A$ and $b = \#B$. Then a^b is the cardinal number defined by

$$a^b = \#(A^B).$$

For notational convenience, we agree that $0^a = 0$ and $a^0 = 1$.

8.4 Theorem For any cardinals a, b, c the following rules hold:

i) $a^{b+c} = a^b a^c$,

ii) $(ab)^c = a^c b^c$,

iii) $(a^b)^c = a^{bc}$.

Proof. Let A, B, C be sets such that

$$a = \#A, \quad b = \#B, \quad c = \#C.$$

(For part (i) of the proof, assume $B \cap C = \varnothing$.)

i) We must show that there exists a bijective function $\sigma : A^{B \cup C} \to A^B \times A^C$.

We define σ as follows: If $f \in A^{B \cup C}$, then $\sigma(f) = (f_{[B]}, f_{[C]})$, where $f_{[B]}$ is the restriction of f to B and $f_{[C]}$ is the restriction of f to C. It is immediate that σ is a function from $A^{B \cup C}$ to $A^B \times A^C$.

σ *is injective*. Suppose $\sigma(f) = \sigma(g)$, where f, $g \in A^{B \cup C}$; then

$$(f_{[B]}, f_{[C]}) = (g_{[B]}, f_{[C]}),$$

that is,

$$\sigma : A^C \times B^C \to (A \times B)^C.$$

Thus, by Theorem 2.15,

$$f(c) = (f_1(c), f_2(c)), \quad \forall c \in C;$$

σ is surjective. For if $(f_1, f_2) \in A^B \times A^C$ then, by Theorem 2.16,

$$\forall c \in C, \quad (f_1(c), f_2(c)) = f(c) = f'(c) = (f_1'(c), f_2'(c)),$$

ii) We will show that there exists a bijective function

$$(f_1, f_2) = (f_1', f_2').$$

We define σ as follows: If $(f_1, f_2) \in A^C \times B^C$, then $\sigma(f_1, f_2)$is the function f defined by

$$f(c) = (f_1(c), f_2(c)), \quad \forall c \in C;$$

certainly $f \in (A \times B)^C$. Now it is immediate that σ is a function from $A^C \times B^C \rightarrow (A \times B)^C$.
σ is injective. For if $f = \sigma(f_1, f_2) = \sigma(f_1, f_2) = f$, then

$$\forall c \in C, \quad (f_1(c), f_2(c)) = f(c) = f'(c) = (f_1'(c), f_2'(c)),$$

so $f_1(c) = f'_1(c)$and $f_2(c) = f'_2(c)$. It follows that $f_1 = f'_1$ and $f'_2 = f'_2$, hence

$$(f_1, f_2) = (f_1', f_2').$$

σ is surjective. For if $f \in (A \times B)^C$, we may define

$$f_1 = \{(x, y) : (x, (y, z)) \in f\}$$

and

$$f_2 = \{(x, z) : (x, (y, z)) \in f\}.$$

It is easily shown that $f_1 \in A^C$, $f_2 \in B^C$, and $f = \sigma(f_1, f_2)$; the details are left to the reader.

iii) We will show that there exists a bijective function $\sigma : (A^B)^C \rightarrow A^{B \times C}$. Note that if $f \in (A^B)^C$ and $c \in C$, then $f(c) \in A^B$; thus, if $b \in B$, then $[f(c)](b) \in A$. Now define $\sigma(f)$to be the function $\hat{f}$ given by

$$\hat{f}(b, c) = [f(c)](b).$$

Certainly, $\sigma(f) = \hat{f} \in A^{B \times C}$. Now it is immediate that σ is a function from $(A^B)^C$ to $A^{B \times C}$.
σ is injective. For if $\hat{f} = \sigma(f) = \sigma(f') = \hat{f}'$, then

$$\forall (b, c) \in B \times C, \quad [f(c)](b) = \hat{f}(b, c) = \hat{f}'(b, c) = [f'(c)](b).$$

Thus $\forall c \in C, f(c) = f'(c)$, so finally, $f = f'$.
σ is surjective. For if $g \in A^{B \times C}$ and $c \in C$, let f_c be defined by

$$f_c(b) = g(b, c), \quad \forall b \in B;$$

Clearly $f_c \in A^B$,Now,if f is given by $f(c) = f_c$, it is easily verified that f is a function from C to A^B; clearly $g = \sigma(f)$.

The finite cardinal numbers are designated, as usual, by the symbols 0, 1, 2, and so forth. It is to be especially noted that 0 is the cardinal number of the empty set, and 1 is the cardinal number of any singleton. The cardinal number of ω—that is, the cardinal number of any denumerable set—is customarily designated by the symbol $\aleph_0{}^0$ (“aleph-null”).

It is useful to distinguish between *finite cardinal numbers*—that is, cardinal numbers of finite sets—and *infinite*,or *transfinite*, *cardinal numbers*, which are the cardinal numbers of infinite sets. We will see, shortly, that infinite cardinals have several properties which do not hold for finite cardinals.

EXERCISES 8.2

1. Prove each of the following, where a is any cardinal number.
   a) $a+0=a$, b) $a0=0$, c) $0^a=0$.
2. Prove each of the following, where a is any cardinal number.
   a) $1a=a$, b) $a^1=a$, c) $1^a=1$.
3. If a, b are arbitrary cardinal numbers, prove that $ab = 0$ if and only if $a = 0$ or $b = 0$.
4. If a, b are arbitrary cardinal numbers, prove that $ab = 1$ if and only if $a = 1$ and $b = 1$.
5. Give a counterexample to the rule: $a + b = a + c \Rightarrow b = c$.
6. Give a counterexample to the rule: $ab = ac \Rightarrow b = c$.
7. If n is a finite cardinal number, use induction to prove that $na = a + a +\ldots+ a$, where the right-hand side of the equality has n terms.
8. If n is a finite cardinal number, use induction to prove that $a^n = aa \ldots a$, where the right-hand side of the equality has n factors.
9. Let a, b be cardinals, and let A, B be sets such that $a = \#A$ and $b = \#B$. Prove that $a + b = \#(A\cup B)+ \#(A\cap B)$.
10. Prove that if a is an infinite cardinal number and n is a finite cardinal number, then $a + n = a$.
11. Prove that if $a + 1 = a$, then a is an infinite cardinal number.
12. If b is an infinite cardinal number, prove that $\aleph_{0_0} + b = b$.
13. Prove: If $A_1 \approx A$ and $B_1 \approx B$ then $A_1 \cup B_1 \approx A \cup B$.
14. Prove: If $A_1 \approx A$ and $B_1 \approx B$ then $A_1 \times B_1 \approx A \times B$.

3 ORDERING OF THE CARDINAL NUMBERS

Since cardinal numbers measure the size of sets, we naturally expect the cardinal number of a smaller set to be “less than” the cardinal number of a larger set. This suggests a natural ordering of the cardinal numbers:

Let a and b be cardinals, and let A and B be sets such that $a = \#A$ and $b = \#B$. The relation $\leqslant$ is defined by

$$a \leqslant b \quad \text{if and only if} \quad A \preccurlyeq B.$$

Note. Clearly, $a \leqslant b$ if and only if $a \preccurlyeq b$. In particular, $a \leqslant b$ if and only if there exists an injective function $f : a \to b$.

Our goal in this section is to show that the relation $\leqslant$ defined above is an order relation among the cardinal numbers, and, in particular, that the class of all the cardinal numbers is well ordered with respect to this relation.

8.5 Theorem (*Schröder-Bernstein*). Let a and b be cardinal numbers; if $a \leqslant b$ and $b \leqslant a$, then $a = b$.

Proof. Suppose $a \leqslant b$ and $b \leqslant a$; if A and B are sets such that $a = \#A$ and $b = \#B$, then $A \preccurlyeq B$ and $B \preccurlyeq A$, that is, there exist injective functions $f : A \to B$ and $g : B \to A$. If $C \subseteq A$, let $(c) = A - \bar{g}[B - \bar{f}\,(C)]$; it is easy to see that if C and D are subsets of A, then

1) $$C \subseteq D \text{ implies } \Delta(C) \subseteq \Delta(D).$$

Indeed,

$$\begin{aligned} C \subseteq D &\Rightarrow \bar{f}(C) \subseteq \bar{f}(D) && \text{(this is half of Theorem 2.29)} \\ &\Rightarrow B - \bar{f}(D) \subseteq B - \bar{f}(C) && \text{by elementary class algebra} \\ &\Rightarrow \bar{g}[B - \bar{f}(D)] \subseteq \bar{g}[B - \bar{f}(C)] \\ &\Rightarrow A - \bar{g}[B - \bar{f}(C)] \subseteq A - \bar{g}[B - \bar{f}(D)]. \end{aligned}$$

Now, let $S = \{B : B \subseteq A \text{ and } B \subseteq \Delta(B)\}$, and let $A_1 = \bigcup_{B \in S} B$. We will prove that $A_1 = \Delta(A_1)$.

i) If $a \in A_1$, then $a \in B$ for some $B \in S$;but $B \subseteq A_1$, so by (1), $(B) \subseteq (A_1)$. Thus we have

$$a \in B \subseteq \Delta(B) \subseteq \Delta(A_1);$$
$$\text{this proves that } A_1 \subseteq \Delta(A_1).$$

ii) We have just shown that $A_1 \subseteq (A_1)$, hence by (1), $\Delta(A_1) \subseteq \Delta[\Delta(A_1)]$,so $\Delta(A_1) \in S$. But A_1 is the union of all the elements of S,so $\Delta(A_1) \subseteq A_1$. Thus, we have proved that $A_1 = (A_1)$, which is the same as

$$A_1 = A - \bar{g}[B - \bar{f}(A_1)].$$

By elementary class algebra (see Exercise 11, Exercise Set 1.3) this gives

2) $$A - A_1 = \bar{g}[B - \bar{f}(A_1)].$$

Now, f and g are injective functions, hence $A_1 \approx \bar{f}(A_1)$and, by (2),

$$B - \bar{f}(A_1) \approx \bar{g}[B - \bar{f}(A_1)] = A - A_1.$$

But $\bar{f}(A_1) \approx A_1$; thus, by 7.9, $A \approx B$.

It is immediate that the relation $\leqslant$ between cardinal numbers is reflexive and transitive; by 8.5 it is antisymmetric, hence it is an order relation. In fact, the cardinal numbers are linearly ordered by $\leqslant$. That is, any two cardinals are comparable.

8.6 Theorem If a and b are cardinal numbers then $a \leqslant b$ or $b \leqslant a$.

Proof. Since a and b are sets, it follows from 5.22 that a and b can be well-ordered. Thus, from 4.62, there exists an injection $a \to b$ or $b \to a$.

8.7 Theorem Every class of cardinal numbers has a least element.

Proof. Let $\mathscr{A}$ be an arbitrary class of cardinal numbers, and let $a \in \mathscr{A}$; if a is the least element of $\mathscr{A}$, we are done; otherwise, let $\mathscr{B}=\{b \in \mathscr{A}: b < a\}$. Using the well-ordering theorem, let us well order a; for each $b \in \mathscr{B}$, let $\varphi(b)$be the least element $x \in a$ such that $b \approx S_x$. Now the set $\{\varphi(b): b \in \mathscr{B}\}$ has a least element $\varphi(d)$because it is a subset of a; we will show that d is the least element of $\mathscr{B}$. Indeed, let b be an arbitrary element of $\mathscr{B}$; $\varphi(d) \leqslant \varphi(b)$, hence $S_{\varphi(d)} \subseteq S_{\varphi(b)}$. Thus we have injective functions

$$d \to S_{\phi(d)} \xrightarrow{\lambda} S_{\phi(b)} \to b$$

(λ is the inclusion function), hence $d \leqslant b$. Thus d is the least element of Let $\mathscr{B}$, hence the least element of $\mathscr{A}$.

We are able to conclude:

8.8 The class of all the cardinal numbers, ordered by $\leqslant$, is well ordered.

The familiar “rules of inequality” apply to the cardinal numbers, as we shall see next.

8.9 Theorem Let a, b be cardinal numbers. Then $a \leqslant b$ if and only if there exists c such that $b = a + c$.

Proof

i) Suppose $b = a + c$; let A, B, C be sets (assume $A \cap C = \varnothing$) such that

$$a = \#A, \quad b = \#B, \quad \text{and} \quad c = \#C.$$

Then there exists a bijective function $f : A \cup C \to B$. Clearly $f_{[A]}$ is an injective function from A to B,so $A \preccurlyeq B$.

ii) Suppose $a \leqslant b$; let A, B be disjoint sets such that

$$a = \#A, \quad b = \#B.$$

There exists an injective function $f : A \to B$; since f is injective, $A \approx \bar{f}$(A),so $a = \#f(A)$. If $C = B - f(A)$ and $c = \#C$, clearly $b = a + c$.

8.10 Theorem Let a, b, c, d be cardinal numbers. If $a \leqslant c$ and $b \leqslant d$, then we have the following:

i) $a + b \leqslant c + d$, ii) $ab \leqslant cd$, iii) $a^b \leqslant c^d$.

Proof. By Theorem 8.8, there exists r, s such that $c = a + r$ and $d = b + s$.

i) $c + d = a + r + b + s = (A+ b)+ (r + s)$, so by 8.8, $a + b \leqslant c + d$.

ii) $cd = (a+ r)(b+ s)= ab + as + rb + rs = ab + (as + rb + rs)$,soby Theorem 8.8, $ab \leqslant cd$.

iii) First we must show that $a^b \leqslant (a+r)^b$; that is, if A, R, B are sets such that $a = \#A$, $b = \#B$ and $r = \#R$, we must show that there exists an injective function $\sigma : A^B \to (A \cup R)^B$. We define σ by

$$\sigma(f) = f, \quad \forall f \in A^B,$$

and not (see 2.4) that a function $f : B \to A$ is also a function $f : B \to A \cup R$. It is immediate that σ is injective, hence $a^b \leqslant (a+r)^b$, that is $a^b \leqslant c^b$. Finally, using part (ii), we have

$$a^b = a^b 1 \leqslant c^b c^s = c^{b+s} = c^d. \ \blacksquare$$

Remark. It is important to note that 8.8 gives us valuable new information on the relation $A \preccurlyeq B$ among sets. Indeed, the following are two immediate consequences of 8.8:

1) If A and B are arbitrary sets, then $A \preccurlyeq B$ or $B \preccurlyeq A$.
2) If $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \approx B$.

Item (2) is especially useful when we need to prove that two sets are in one-to-one correspondence, for it is now sufficient to show that there is an injective function from A to B and an injective function from B to A (alternatively, a surjective function from A to B and a surjective function from B to A).

EXERCISES 8.3

1. Prove that if A and B are arbitrary sets, then $A \preccurlyeq B$ or $B \preccurlyeq A$.
2. Prove that is $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \approx B$.
3. Prove the following, where a, b, c, d are cardinal numbers.

   a) If $a^c < b^c$, then $a < b$.

   b) If $a^c < a^d$, then $c < d$.
4. Prove that if $a + b = c$, then $(r+s)^c \geq r^a s^b$.
5. Prove that there exists a strictly increasing sequence $a_1 < a_2 < \ldots$ of cardinal numbers, each with the property $a_i^{\aleph_0 0} = a_i$.[*Hint*: Take $a_1 = \aleph_{00}^{\aleph_0 0}$; then take $a_2 = 2^{a 1 \aleph_0 0}$,$a3 = 2^{a 2 \aleph_0 0}$, etc.]
6. Let A be a denumerable set. Prove each of the following:

   a) $A^A \subseteq \mathscr{P}(A \times A)$. Conclude that $A^A \preccurlyeq \mathscr{P}(A \times A)$, hence $A^A \preccurlyeq \mathscr{P}(A)$.

   b) Verify that the function φ given by: $\varphi(f)$= range f, $\forall f \in A^A$, is a surjective function $A^A \to \mathscr{P}(A) - \varphi$. Conclude that $\mathscr{P}(A) \preccurlyeq A^A$.

   c) $A^A \approx (\mathscr{P}(A)$; that is, $A^A \approx 2^A$. Conclude that $\aleph_{00}^{\aleph_0 0} = 2^{\aleph_0 0}$.
7. Use the argument outlined in the preceding problem to prove that the set of all injective functions $A \to A$ is equipotent with 2^A.

4 SPECIAL PROPERTIES OF INFINITE CARDINAL NUMBERS

A few remarkable arithmetic rules hold exclusively for *infinite* cardinals. As a result of these rules, the arithmetic of infinite cardinal numbers is a very simple matter.

8.11 Theorem If a is an infinite cardinal number, then $aa = a$.

Proof. Let A be a set such that $a = \#A$. Since A is infinite, A has a denumerable subset D. By Corollary 7.21, $D \approx D \times D$; that is, there exists a bijective function $\varphi : D \to D \times D$. Now let A be the set of all pairs $(B, f\,)$which satisfy the following conditions:

i) B is a subset of A and f is a bijective function from B to $B \times B$.

ii) $D \subseteq B$.

iii) $\varphi \subseteq f$.

We order $\mathscr{A}$ by the relation $(B_1, f_1)\leqslant (B_2, f_2)$ iff $B_1 \subseteq B_2$ and $f_1 \subseteq f_2$. $\mathscr{A}$ is nonempty, for $(D, \varphi) \in A$. Now it is easy to verify that $\mathscr{A}$ satisfies the hypotheses of Zorn's Lemma (the details are left as an exercise for the reader). Thus $\mathscr{A}$ has a maximal element (C, g); it remains only to show that $\#C = a$. We will prove this by contradiction—assuming that $\#C < a$ and proving this to be impossible.

Let $b = \#C$ and assume that $b < a$. Since $C \times C \approx C$, it follows that $bb = b$; furthermore,

$$b = 0 + b \leqslant b + b$$

and

$$b + b = 1b + 1b = (1 + 1)b \leqslant bb = b;$$

hence $b = b + b$. Now let $d = \#(A - C)$; C and $A - C$ are disjoint, so

$$a = \#A = \#(A - C) + \#C = d + b.$$

We note that $b< d$, for $d \leqslant b$ implies that

$$a = d + b \leqslant b + b = b,$$

which would contradict our assumption that $b < a$. From $b < d$ it follows that $A - C$ has a subset E such that $\#E = b$.

Now

$$(C \cup E) \times (C \cup E) = (C \times C) \cup (C \times E) \cup (E \times C) \cup (E \times E),$$

where $C \times C$, $C \times E$, $E \times C$, $E \times E$ are mutually disjoint sets, each of which has the cardinal $bb = b$. Thus

$$\#[(C \times E) \cup (E \times C) \cup (E \times E)] = b + b + b = (b + b) + b = b + b = b,$$

hence there exists a bijective function

$$h\colon E \to [(C \times E) \cup (E \times C) \cup (E \times E)].$$

It follows by 7.8 that $g \cup h$ is a bijective function from $C \cup E$ to

$$(C \times C) \cup [(C \times E) \cup (E \times C) \cup (E \times E)] = (C \cup E) \times (C \cup E),$$

hence $(C \cup E, g \cup h) > (C, g)$, which is impossible because $(C, g)$is a maximal element of $\mathscr{A}$.

The assumption that $b < a$ has led to a contradiction; thus $b = a$,so $aa = a$.

8.12 Corollary Let a and b be cardinals, where a is infinite and $b \neq 0$. If $b \leqslant a$, then $ab = a$.

Proof. Since $b \geqslant 1$, thus $a = a1 \leqslant ab$;but $ab \leqslant aa = a$, hence $ab = a$.

8.13 Corollary If a is an infinite cardinal, $a + a = a$.

Proof. We have $a = 1a \leqslant 2a \leqslant aa = a$; but $2a = (1 + 1)a = a + a$, so $a + a = a$.

8.14 Corollary Let a and b be cardinals, where a is infinite. If $b \leqslant a$, then $a + b = a$.

Proof. We have $a = a + 0 \leqslant a + b$;but $a + b \leqslant a + a = a$,so $a + b = a$.

8.15 Corollary Let a and b be infinite cardinal numbers. Then

$$a + b = ab = \max\{a, b\}.$$

8.16 Theorem Let $a > 1$ be a cardinal number and let b be an infinite cardinal number. If $a \leqslant b$, then $a^b = 2^b$.

Proof. By 7.4, $a < 2^a$,so $a^b \leqslant (2^a)^b = 2^{ab}$. But by Corollary 8.12 $ab = b$, so $a^b \leqslant 2^b$. On the other hand, $2 \leqslant a$, so $2^b \leqslant a^b$. Consequently $a^b = 2^b$.

Remark. Theorem 8.11 and its corollaries can be interpreted very profitably in terms of sets and the relation $A \preccurlyeq B$ among sets. For example, Theorem 8.11 tells us that if A is an infinite set, then $A \approx A \times A$. This has an interesting consequence: $A \times A$ has a partition $\{B_x\}_{x \in A}$ where $B_x = \{(x, y) : y \in A\}$. Hence the bijective function from $A \times A$ to A induces a corresponding partition $\{C_x\}_{x \in A}$ of A, where A is the index set and each member of the partition is equipotent with A.

EXERCISES 8.4

1. If a is an infinite cardinal number and $a \leqslant bc$, prove that $a \leqslant b$ or $a \leqslant c$.
2. Let a be a cardinal number > 1, and let b be an infinite cardinal number. Prove that if $a = a^b$, then $b < a$.
3. An infinite cardinal number a is said to be *dominant* if it satisfies the following condition: if b and c are cardinal numbers such that $b < a$ and $c < a$, then $b^c < a$. Prove that a is a dominant cardinal number if and only if $d < a \Rightarrow 2^d <$ a.
4. If a, c, and d are arbitrary cardinal numbers and b is an infinite cardinal number, prove that

$$a + b \leqslant c + d \Rightarrow a \leqslant c \text{ or } b \leqslant d.$$

5. Let a, b, c, d be cardinal numbers. Prove that if $a < b$ and $c < d$, then $ac < bd$ and $a + c < b + d$. [*Hint*: For the case where b and d are both finite, this result has been proven in Exercises 5 and 6, Exercise Set 6.4. Ignore this case, and assume that b is infinite or d is infinite (this assumption includes three cases).]

In Exercises 6 through 8, a, b, and c are arbitrary cardinals. For each of these problems, the case where a, b, and c are all finite has been considered in Chapter 6. Ignore this case, and treat the remaining cases.

6. Prove that if $a + a = a + b$, then $a \geqslant b$.
7. Prove that if $a + b < a + c$, then $b < c$.
8. Prove that if $ab < ac$, then $b < c$.

5 INFINITE SUMS AND PRODUCTS OF CARDINAL NUMBERS

Early in this book we spoke of the union of two classes; later we extended this notion by defining the union of an arbitrary family of classes. Similarly, we introduced the Cartesian product of two classes and later generalized this to the product of a family of classes. In both cases, extending our original definition seemed like a perfectly natural thing to do, for the intuitive concepts of union and product can be applied as easily to a family of classes as to a pair of classes. The same holds true for the process of adding and multiplying cardinal numbers; they lend themselves to the following obvious generalization.

Let $\{a_i\}_{i \in I}$ be a family of cardinal numbers; let $\{A_i\}_{i \in I}$ be a family of disjoint sets such that $a_i = \#A_i$ for each $i \in I$. Then $\sum_{i \in I} a_i$ is the cardinal number defined by

$$\sum_{i \in I} a_i = \#\Big(\bigcup_{i \in I} A_i\Big).$$

Let $\{a_i\}_{i \in I}$ be a family of cardinal numbers, and let $\{A_i\}_{i \in I}$ be a family of sets such that $a_i = \#A_i$ for each $i \in I$. Then $\bigtimes_{i \in I} a_i$ is the cardinal number defined by

$$\bigtimes_{i \in I} a_i = \#\Big(\prod_{i \in I} A_i\Big).$$

In elementary arithmetic we learn that ab is the result of "adding a to itself b times," and that a^b is the result of "multiplying a by itself b times." It is useful to know that this holds true for all cardinal numbers a and b.

8.17 Theorem Let a and b be cardinal numbers, and let I be a set such that $b = \#I$. If $a = a_i$, $\forall i \in I$, then

i) $ab = \sum_{i \in I} a_i$, and

ii) $a^b = \bigtimes_{i \in I} a_i$.

Proof

i) Let $\{A_i\}_{i\in I}$ be a family of disjoint sets such that $a = a_i = \#A_i$ for each $i \in I$, and let A be a set such that $a = \#A$. Since $A_i \approx A$ for each $i \in I$, there exists a family $\{f_i : A \to A_i\}_{i\in I}$ of bijective functions. We define

$$f : A \times I \to \Big(\bigcup_{i\in I} A_i\Big)$$

by

$$f(x,i) = f_i(x);$$

it is elementary to verify that f is bijective. Thus

$$A \times I \approx \Big(\bigcup_{i\in I} A_i\Big);$$

that is,

$$ab = \sum_{i\in I} a_i .$$

ii) We wish to show that $A^I \approx \prod_{i\in I} A_i$, where $A_i = A$ for each $i \in I$. But a glance at the definitions of A^I and $\prod_{i\in I} A_i$ $\mathrm{A_i}$ (where $A_i = A$, $\forall i \in I$) will reveal that they both refer to the same set—the set of all functions from I to A.

Theorem 8.9 has the following analogue for infinite sums and products.

8.18 Theorem Let $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ be families of cardinal numbers. If $a_i \leqslant b_i$ for each $i \in I$, then

$$\text{i)}\ \sum_{i\in I} a_i \leqslant \sum_{i\in I} b_i ,$$

$$\text{ii)}\ \bigtimes_{i\in I} a_i \leqslant \bigtimes_{i\in I} b_i .$$

Proof

i) Let $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ be a families of disjoint sets such that $a_i = \#A_i$ and $b_i = \#B_i$ for each $i \in I$. Since $a_i \leqslant b_i$ for every $i \in I$, there exists a family $\{f_i : A_i \to B_i\}_{i\in I}$ of injective functions. It is easy to verify that$f = \bigcup_{i\in I} f_i$ is an injective function from $\bigcup_{i\in I} A_i$ to $\bigcup_{i\in I} B_i$. (The details are left as an exercise for the reader.)

ii) Given the family $\{f_i : A_i \to B_i\}_{i\in I}$ introduced above, we define a function :

$$f: \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

as follows: if $x \in \prod_{i\in I} A_i$ and $y \in \prod_{i\in I} B_i$, then

$$f(x) = y \quad \text{iff} \quad f_i(x_i) = y_i \quad \forall i \in I.$$

We verify that *f* is injective: If *f* (*u*)= *f(v)*, then

$$f_i(u_i) = f_i(v_i), \quad \forall i \in I.$$

But each f_i is injective, so $u_i = v_i$ for every $i \in I$; hence $u = v$.

Theorem 8.17 and 8.18 have the following useful corollary.

8.19 Corollary Let $\{a_i : i \in I\}$ be a set of cardinal numbers, and let *b* and *c* be cardinal numbers. If $a_i \leqslant b$ for each $i \in I$ and if $\#I = c$, then

i) $\sum_{i\in I} a_i \leqslant bc$, and

ii) $\bigtimes_{i\in I} a_i \leqslant b^c$.

The proof, which follows immediately from 8.17 and 8.18, is left as an exercise for the reader.

EXERCISES 8.5

1. Prove that1. Prove that $\bigtimes_{i\in I} a_i = 0$ if and only if $a_i = 0$ for some $i \in I$.
2. Suppose $ai \leqslant a$, ∀i ∈ *I* , and $\#I \leqslant a$, where *a* is some fixed cardinal. Prove that $\sum_{i\in I} a_i \leqslant a$, [Hint: Use Theorems 8.17 and 8.18.]
3. Suppose $ai \leqslant$ a, ∀i ∈ *I* , and #I $\leqslant$ a. Prove that $\bigtimes_{i\in I} a_i \leqslant 2^a$.
4. Let $\{a_i\}_{i\in I}$ be a set of a cardinal numbers, and suppose there is no greatest element in this set. Prove that $\forall j \in I$, $a_j < \sum_{i\in I} a_i$
5. Prove that $a \bullet \cdot \sum_{i\in I} b_i = \sum_{i\in I} ab_i$.
6. Use Theorem 8.18 to justify each of the following. (Each sum is understood to have $\aleph_{00}$ terms.)

   a) $1 + 2 + 3 + \ldots = \aleph_{00}$, b) $n + n + \ldots = \aleph_{00}$, c) $\aleph_0 0 + \aleph_0 0 + \cdots = \aleph_0 0$.
7. Let $f : A \to B$ be a surjective function, where *B* is an infinite set. If, $\forall y \in B$, $f^{-1}(y)$is finite or denumerable, prove that $A \approx B$.
8. Let *A* be an infinite set, and let *F*(*A*) designate the family of all finite subsets of *A*. Prove that *F* (*A*) ≈ *A*.[*Hint*: For each $n \in \omega$, let F_n designate the family of *n*-element subsets of *A*. There exists an obvious surjective function from A^n to F_n; there are $\aleph_{00}$ set F_n.]

9. If $\{C_i\}_{i\in I}$ is a family of sets, prove that # $\#(\bigcup_{i\in I} C_i) \leqslant \sum_{i\in I} \#C_i$.

10. Let $\{a_i\}_{i\in I}$ and $\{a_i\}_{i\in J}$ be families of cardinal numbers. Prove that $\sum_{i\in I} a_i \leqslant \sum_{i\in I\cup J} a_i$.

9
Arithmetic of the Ordinal Numbers

1 INTRODUCTION

In elementary school we learn that there are cardinal numbers and ordinal numbers. The cardinal numbers, we are told, are the "counting" numbers: 1, 2, 3, and so on; the ordinal numbers are the "ranking" numbers: first, second, third, etc. The distinction may appear to be somewhat pedantic, for the natural numbers serve in both capacities, as ordinals and as cardinals, and there is no need in elementary arithmetic to differentiate between the two. However, one of the unexpected discoveries of modern set theory is that, just as the infinite cardinals behave differently from the finite ones, so the infinite ordinals exhibit a strikingly different behaviour from the cardinals. It is the purpose of this chapter to introduce the ordinal numbers and explore their properties—especially those of the infinite ordinals.

The dichotomy between cardinal and ordinal, from the scholastic point of view, arises from two different ways of *using* the natural numbers. In their role as cardinals, the natural numbers measure the "size," or power, of sets; in their role as ordinals, they serve to designate the rank, or position, of an object in a linearly ordered array. We will use this insight—although it is somewhat outdated—as the starting point of our discussion.

When we speak of ranking elements in some order, what kind of order do we have in mind? There must be a first element, a second element, and so on—in other words, the order is that of the natural numbers, which is a *well-ordering*. Now the reader should note that the general notion of well-ordering is an extension of the order of the natural numbers. Every infinite well-ordered set has a first element, a second element, and—for each natural number *n*—an *n*th element; but it may also have elements which are "beyond the reach" of the finite ordinals. Thus the set

$$\{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}$$

has a first element x_1, a second element x_2, and so forth; but y_1, for example, though it has a perfectly well-defined "position" in the set, cannot be described as the "*n*th element" for any finite *n*.

Situations of this kind arise frequently in almost every branch of mathematics. For example, on page 141 we defined a sequence of sets by these conditions:
$\omega = K_1$, $\mathscr{P}(K_1) = K_2$, and so on; $\bigcup_{i \in \omega} K_i = L_1$, $\mathscr{P}(L_1) = L_2$, etc. Continuing in this manner, we get the well-ordered family of sets

$$\{K_1 \prec K_2 \prec K_3 \prec \cdots \prec L_1 \prec L_2 \prec L_3 \prec \cdots \prec M_1 \prec M_2 \prec M_3 \prec \cdots\}.$$

Now K_1 is the first element of this family, and, in general, K_n is the *n* th element; but what of (say) L_1? Its position in the family is unambiguous: L_1 immediately follows *all* of the sets K_i ; yet classical mathematics has not provided us with any ordinal number to describe the position of L_1.

Thus, as in our study of cardinal numbers, we are led to ask an intriguing question: Can we find a way of extending our system of ordinal numbers so as to create a set of standards for designating the position of any element in any well-ordered set? The answer, once again, is “yes;” we can generalize the concept of ordinal number with such remarkable ease that no barrier, either logical or intuitive, seems to separate the finite ordinals from the infinite ones.

We will approach the ordinals in much the same way that we approached the cardinals. We will begin by defining a relation of “having the same ordinal number,” and later define the ordinals, essentially, to be representative of the distinct classes induced by this relation.

If A and B are well-ordered sets, and if $x \in A$ and $y \in B$, then to say that “x has the same rank as y” (for example, x is the third element of A and y is the third element of B) is the same as saying that the initial segment S_x is isomorphic with the initial segment S_y. To say that “x has a lower rank than y” is the same as saying that S_x is isomorphic with an initial segment of S_y. The reader should stop here until he has thoroughly understood this fact, for it is the point of departure for achieving an understanding of the modern approach to the ordinal numbers. Isomorphism plays the same role in the study of ordinal numbers that one-to-one correspondence plays in the study of cardinal numbers.

An important warning needs to be given here. The alert reader may question the necessity of introducing the concept of isomorphism. After all, he may ask, why not say that x and y have the same rank if and only if S_x is *equipotent* with S_y? Surely if ten elements precede x and ten elements precede y, then x and y are both eleventh in their class. This is true when we are dealing with finite rank, but untrue in the case of infinite rank; a simple example will convince the reader. In the set

$$\{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\},$$

both y_1 and y_2 are preceded by a denumerable number of elements—that is, S_{y1} is equipotent with S_{y2}—yet y_2 clearly follows y_1.

When we say that x has the same rank as y, or x has a lower rank than y, we are only *apparently* speaking of x and y; actually, we are comparing the initial segments S_x and S_y. Hence we lose nothing if we confine our attention to the study of initial segments of well-ordered sets. But we can go a step further: An initial segment (of a well-ordered set) is a well-ordered set, and conversely, every well-ordered set A is an initial segment (adjoin a last element x to A—then A is S_x). Hence the study of ordinality is, essentially, the study of well-ordered sets.

Motivated by the foregoing remarks, we introduce the following definitions:

Let A and B be well-ordered sets. We say that A and B are *similar* (or have the *same ordinality*) if A is isomorphic with B; we write $A \approxeq B$. If A is isomorphic with an initial segment of B, we say that B is a *continuation* of A, or A has a *lower ordinality* than B, and we write $A -\!\!< B$.

It follows from Theorem 4.62 that if A and B are any two well-ordered sets, then $A \approxeq B$, or $A -\!\!< B$, or $B -\!\!< A$.

In conclusion, to say that x has the same rank as y is the same as saying that $S_x \approxeq S_y$, and to say that x has a lower rank than y is the same as saying that $S_x -\!\!< S_y$. Thus we have completely captured—and formalized—the intuitive concept of “rank,” and have extended it beyond the unnatural confines of finite ordinality.

As for the ordinal numbers, we simply imitate the procedure we followed for the cardinals by introducing the

A13 Axiom of Ordinality There is a class OR of well-ordered sets, called *ordinal numbers*, with the following properties:

O1 If A is any well-ordered set, there exists an ordinal number α such that $A \approxeq \alpha$.
O2 If A is a well-ordered set and α, β are ordinal numbers, then $A \approxeq \alpha$ and $A \approxeq \beta \Rightarrow \alpha = \beta$.

We will add the Axiom of Ordinality to our list of axioms for set theory—but only on a provisional basis, for in the last section of this chapter we will describe a method for *constructing* sets with properties 01 and 02; those sets will then serve the purpose of ordinal numbers.

It is worth noting, incidentally, that the *class of all the ordinal numbers* is a proper class. Indeed, let OR be the class of all the ordinal numbers, and suppose OR is a set; from this assumption we will derive a contradiction. Let $A = \mathscr{P}(B)$, where $B = \bigcup_{\alpha \in \mathrm{OR}} \alpha$; since each ordinal number α is a set and OR (under our assumption) is a set, it follows by Axioms A6 and A7 that B, and therefore A, are sets. Let us well-order A; by O1, there is an ordinal number α such that $\alpha \approxeq A$. But $\alpha \in$ OR, hence $\alpha \subseteq B$, so by 7.2, α cannot be equipotent with $A = \mathscr{P}(B)$. This contradiction proves that OR is a proper class.

2 OPERATIONS ON ORDINAL NUMBERS

Following our “naive” introduction to ordinal numbers in the preceding section, we now proceed to study the ordinals from a formal point of view. We will henceforth consider the ordinals to be the objects defined by Conditions O1 and O2. The reader should adjust his thinking accordingly; he should cease thinking of ordinals as “symbols for designating rank,” and begin to think of them as certain *well-ordered sets*.

9.1 Definition If A is a well-ordered set, α is an ordinal number, and $A \approxeq \alpha$, then we say that *α is the ordinal number of A*. We denote this by writing

$$\alpha = \circledbslash A.$$

Now Conditions O1 and O2 can be conveniently restated as follows:

O1 If A is a well-ordered set, there exists an ordinal number α such that $\alpha = \circledbslash A$.

O2 If A is a well-ordered set and α, β are ordinal numbers, then $\alpha = \circledbslash A$ and $\beta = \circledbslash A \Rightarrow \alpha = \beta$.

9.2 Lemma If α and β are ordinal numbers and $\alpha \approxeq \beta$, then $\alpha = \beta$.

The proof is an immediate consequence of 02.

9.3 Lemma If $A \approxeq B$, then $\circledbslash A = \circledbslash B$.

The proof is analogous to that of Lemma 8.2.

Before defining the addition and multiplication of ordinal numbers, we need to introduce two new operations on well-ordered sets.

9.4 Definition Let A and B be disjoint, well-ordered sets. $A \oplus B$, called the *ordinal sum* of A and B, is the set $A \bigcup B$ ordered as follows. If $x, y \bigcup A \bigcup B$, then $x \leqslant y$ if and only if

i) $x \in A$ and $y \in A$ and $x \leqslant y$ in A, or

ii) $x \in B$ and $y \in B$ and $x \leqslant y$ in B, or

iii) $x \in A$ and $y \in B$.

Thus, in $A \oplus B$, the elements of A are ordered as before, the elements of B are ordered as before, and every element of B is greater than every element of A.

Having defined the ordinal sum of two well-ordered sets, it is natural to define the ordinal sum of an arbitrary family of well-ordered sets.

9.5 Definition Let I be a set, let $\{A_i\}_{i\in I}$ be a family of disjoint well-ordered sets and let the index set I be well-ordered.

$\underset{i\in I}{\mathsf{S}} A_i$, called the *ordinal sum* of the family $\{A_i\}_{i\in I}$, is the set $\bigcup_{i\in I} A_i$ ordered in the following way: if $x, y \in \bigcup_{i\in I} A_i$, then $x \leqslant y$ if and only if

i) for some $i \in I$, $x \in A_i$ and $y \in A_i$ and $x \leqslant y$ in A_i, or

ii) $x \in A_i$ and $y \in A_j$ and $i < j$.

Thus, in $\underset{i\in I}{\mathsf{S}} A_i$ A_i, each set A_i is ordered as before, and, for $i < j$, every element of A_i is less than every element of A_j.

An easy step leads us, now, to the notion of ordinal product. To put it simply, the product $A \odot B$ is the result of “adding A to itself B times.” More precisely, if $\{A_i\}_{i\in B}$ is a family of disjoint, well-ordered sets, indexed by B, where each A_i is similar to A, then $A \odot B$ is the set $\underset{i\in B}{\mathsf{S}} A_i$. The only remaining difficulty is to produce the family $\{A_i\}_{i \in B}$. To do so is easy enough: for each $i \in B$, we define A_i to be the set $\{(x, i)\colon x \in A\}$—that is, $A_i = A \times \{i\}$. But a happy thought strikes us now, as we realize that the set $\underset{i\in B}{\mathsf{S}} A_i$ is none other than the Cartesian product $A \times B$ ordered by the antilexicographic ordering (Definition 4.2). We exploit this fortunate coincidence to give the following elegant definition of ordinal product:

9.6 Definition Let A and B be well-ordered sets. Then $A \odot B$, called the *ordinal product* of A and B, is the set $A \times B$ ordered by the antilexicographic ordering.

9.7 Example Let $A = \{a, b, c\}$ be well ordered as follows: $a < b < c$. Let $B = \{1 < 2 < 3\}$ be well ordered as follows: $1 < 2 < 3$. Then $A \odot B$ is the set $A \times B$ well ordered as follows (see Fig. 9.1):

$$(a, 1) < (b, 1) < (c, 1) < (a, 2) < (b, 2) < (c, 2) < (a, 3) < (b, 3) < (c, 3).$$

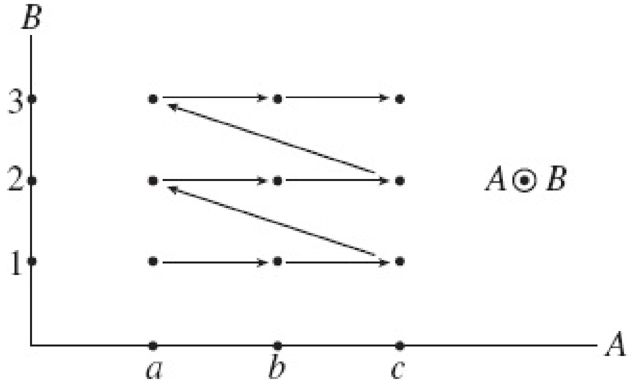


Fig. 9.1

Now, back to the ordinal numbers.

9.8 Definition Let α and β be ordinal numbers, and let A and B be disjoint, well-ordered sets such that $\alpha = \circledbslash A$ and $\beta = \circledbslash B$. We define the sum $\alpha + \beta$ to be the ordinal number given by

$$\alpha + \beta = \circledbslash(A \oplus B).$$

Let α and β be ordinal numbers, and let A and B be well-ordered sets such that $\alpha = \circledbslash A$ and $\beta = \circledbslash B$. We define the product $\alpha\beta$ to be the ordinal number given by

$$\alpha\beta = \circledbslash(A \odot B).$$

In order to tie down the definition of ordinal addition and multiplication, it must be shown that these operations are well-defined. That is, if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$ and $\alpha_1\beta_1 = \alpha_2\beta_2$. This is easy to prove and left as an exercise at the end of this Section.

The elementary properties of ordinal addition and multiplication are given in the following theorem.

9.9 Theorem Let α, β and γ be ordinal numbers. Then

i) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$,

ii) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$,

iii) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

Proof. Let A, B, and C be disjoint, well-ordered sets such that $\alpha = \circledbslash A$, $\beta = \circledbslash B$, and $\gamma = \circledbslash C$.

i) We must show that

$$A \oplus (B \oplus C) \approxeq (A \oplus B) \oplus C.$$

But it follows immediately from our definition of ordinal sums that both $A \oplus (B \oplus C)$ and $(A \oplus B) \oplus C$ designate the set $A \cup B \cup C$ with the following order: If x and y are both in A, both in B, or both in C, they are ordered according to their order in A, B, or C respectively; furthermore, every element of C is greater than every element of B, and every element of B is greater than every element of A.

ii) We must show that

$$A \odot (B \odot C) \approxeq (A \odot B) \odot C.$$

We have seen earlier that the function

$$f(x, (y, z)) = ((x, y), z)$$

is a one-to-one correspondence between $A \times (B \times C)$ and $(A \times B) \times C$. In order to establish that f is an isomorphism, we need simply show that

$$(x, (y, z)) \leqslant (x', (y', z'))$$

if and only if

$$((x, y), z) \leqslant ((x', y'), z').$$

The details, which follow immediately from the definition of ordinal product, are left as an exercise for the reader.

iii) We must show that

$$A \odot (B \oplus C) \approxeq (A \odot B) \oplus (A \odot C).$$

Both $A \odot (B \oplus C)$ and $(A \odot B) \oplus (A \odot C)$ designate the same set,

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

with certain orderings; it is easy to show that the two orderings are the same. The details are left as an exercise for the reader.

As usual, 0 is the ordinal number of the empty set, and 1 is the ordinal number of any singleton. An ordinal number μ is said to be *finite* if μ is similar to a natural number n; if μ is not a finite ordinal, then μ is called an *infinite*, or *transfinite*, ordinal. It is customary to designate the ordinal number of ω by means of the symbol ω.

It is most important to note that *addition and multiplication of ordinal numbers are not commutative*. Two simple examples will suffice to establish this fact. First, let us take addition. If we compare $\omega + 1$ with $1 + \omega$, we observe that $1 + \omega$ is similar to ω, whereas $\omega + 1$ is not (it has a last element!); thus $\omega + 1 \neq 1 + \omega$. Next, let us take multiplication. We observe that

$$2\omega = \{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2), (1, 2), \ldots\}$$

and that

$$\omega 2 = \{(0, 0), (1, 0), (2, 0), \ldots, (0, 1), (1, 1), (2, 1), \ldots\};$$

these sets are obviously not isomorphic, hence $\omega 2 \neq 2\omega$.

We note also that the “*right distributive law* “ $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$ *does not hold*. For example,

$$(1 + 1)\omega = 2\omega,$$

whereas

$$1\omega + 1\omega = \omega + \omega = \omega(1 + 1) = \omega 2,$$

and we noted in the preceding paragraph that $2\omega \neq \omega 2$; thus

$$(1+1)\omega \neq 1\omega + 1\omega.$$

EXERCISES 9.2

1. Let A_1, A_2, B_1, B_2 be well-ordered sets. Prove that if $A_1 \approxeq A_2$ and $B_1 \approxeq B_2$, then

   a) $A_1 \oplus B_1 \approxeq A_2 \oplus B_2$, and

   b) $A_1 \odot B_1 \approxeq A_2 \odot B_2$.
2. If A and B are well-ordered sets, prove that $A \oplus B$ and $A \odot B$ are well-ordered sets.
3. If α is an ordinal number, prove that $1 + \alpha = \alpha$ iff α is an infinite ordinal.
4. Let α and β be nonzero ordinal numbers. Prove that if $\alpha + \beta = \omega$, then α is a finite ordinal number (that is, similar to a natural number) and $\beta = \omega$. Now assume $\beta \neq 1$. Prove that if $\alpha\beta = \omega$, then α is finite and $\beta = \omega$. [*Hint*: Consider the well-ordered sets $A \oplus B$ and $A \odot B$, where $\alpha = \obslash A$ and $\beta = \obslash B$.]
5. Prove each of the following, where μ designates a finite ordinal.

   a) $\mu + \omega = \omega$,

   b) $\mu\omega = \omega$,

   c) If α is an infinite ordinal, then $\mu + \alpha = \alpha$.
6. Prove that $(\omega + \omega)\omega = \omega\omega$.
7. Give a counterexample to the (false) rule

$$\alpha + \gamma = \beta + \gamma \Rightarrow \alpha = \beta.$$

8. Prove the following, for every ordinal number α.

   a) $0 + \alpha = \alpha + 0 = \alpha$, b) $\alpha 0 = 0\alpha = 0$, c) $\alpha 1 = 1\alpha = \alpha$.
9. Prove that $\alpha\beta = 0$ if and only if $\alpha = 0$ or $\beta = 0$.
10. Prove each of the following, where μ, ν, π are finite ordinals.

    a) $m \approxeq \mu$ iff $m \approx \mu$ (where $m \in \omega$), b) $\mu + \nu = \nu + \mu$,

    c) $\mu\nu = \nu\mu$, d) $(\mu + \nu)\pi = \mu\pi + \nu\pi$.
11. Give a complete proof of the isomorphism [Theorem 9.9(ii)]

$$A \odot (B \odot C) \approxeq (A \odot B) \odot C.$$

12. Give a complete proof of the isomorphism [Theorem 9.9(iii)]

$$A \odot (B \oplus C) \approxeq (A \odot B) \oplus (A \odot C).$$

3 ORDERING OF THE ORDINAL NUMBERS

In the introduction to this chapter, we spoke of comparing the ordinality of well-ordered sets. If A and B are well-ordered sets, we say that A has a *lower ordinality* than B, (in symbols $A \prec B$) if A is isomorphic with an initial segment of B. It is convenient now to add: the ordinality of A is *less than or equal to* the ordinality of B if and only if A is isomorphic with B or an initial segment of B; in this case, we write $A \preccurlyeq B$. This is the same as saying that there exists an injective, order-preserving function from A to B, whose range is a section of B. (See 4.48 and 4.56.)

9.10 Lemma Let A and B be well-ordered sets. $A \preccurlyeq B$ if there exists an injective, order preserving function $f : A \to B$.

Proof. If $A \preccurlyeq B$ then clearly there exists an injective, order preserving function from A to B.

Conversely, suppose there exists an injective, order preserving function $f : A \to B$. If $C = \bar{f}(A)$, then $f : A \to C$ is an isomorphism. By 4.63, there exists an isomorphism $g : C \to D$, where D is B or an initial segment of B. Now $g \circ f : A \to D$ is an isomorphism, hence $A \preccurlyeq B$. ■

9.11 Corollary If $A \preccurlyeq B$ and $B \preccurlyeq C$ then $A \preccurlyeq C$.

Proof. Clearly the composite of two injective, order preserving functions is injective and order preserving. ■

If A and B are well-ordered sets and A has a lower ordinality than B, we quite naturally expect the ordinal number of A to be “less than” the ordinal number of B. Accordingly, we define the “natural” ordering of the ordinal numbers as follows.

9.12 Definition Let α and β be ordinals, and let A and B be well-ordered sets such that $\alpha = \obslash A$ and $\beta = \obslash B$. The relation $\leqslant$ is defined by

$$\alpha \leqslant \beta \quad \text{if and only if} \quad A \preccurlyeq B.$$

We note that $\alpha \leqslant B$ if and only if $\alpha \preccurlyeq \beta$.

The relation $\leqslant$ which we have just defined is obviously reflexive; it is antisymmetric by Lemma 4.61; it is transitive by Lemma 9.10; hence it is an order relation among the ordinal numbers.

Next, we are able to show that any two ordinal numbers are comparable. That is, if α and β are ordinals, then $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$. This fact follows immediately from 4.62. Moreover:

9.13 Theorem Every nonempty class of ordinal numbers has a least element.

Proof. Let $\mathcal{O}$ be a nonempty class of ordinal numbers, and let α be an arbitrary element of $\mathcal{O}$. If α is the least element of $\mathcal{O}$, we are done; otherwise, let $\mathscr{B} = \{\beta \in \mathcal{O} : \beta < \alpha\}$. It follows from our definition of the relation $<$ that every $\beta \in \mathscr{B}$ is similar to an initial segment of α. For each $\beta \in \mathscr{B}$, let $\phi(\beta)$ be the least element $x \in \alpha$ such that $\beta \approxeq S_x$. Now the set $\{\phi(\beta) : \beta \in \mathscr{B}\}$ has a least element $\phi(\delta)$ because it is a subset of α. We will show that δ is the least element of $\mathscr{B}$.

Indeed, let $\beta \in \mathscr{B}$; then $\phi(\delta) \leqslant \phi(\beta)$, hence $S_{\phi(\delta)} \subseteq S_{\phi(\beta)}$, so by 4.63, $S_{\phi(\delta)} \preccurlyeq S_{\phi(\beta)}$. Thus we have

$$\delta \approxeq S_{\phi(\delta)} \preccurlyeq S_{\phi(\beta)} \approxeq \beta,$$

so, by 9.10, $\delta \leqslant \beta$. ■

Thus, **the class of all the ordinal numbers is well ordered**.

9.14 Theorem

i) If $\beta > 0$, then $\alpha < \alpha + \beta$.

ii) $\beta \leqslant \alpha + \beta$.

Proof

i) Let A and B be well-ordered sets such that $\alpha = \circledbslash A$ and $\beta = \circledbslash B$. If b is the least element of B, then clearly A is the initial segment S_b of $A \oplus B$. Thus $A \prec A \oplus B$, so $\alpha < \alpha + \beta$.

ii) $B \subseteq A \oplus B$, hence by 4.63, B is isomorphic with $A \oplus B$ or an initial segment of $A \oplus B$. Thus $B \preccurlyeq A \oplus B$, so $\beta \leqslant \alpha + \beta$. ■

9.15 Theorem Let α and β be ordinals such that $\alpha < \beta$. Then there exists a unique ordinal $\gamma > 0$ such that $\alpha + \gamma = \beta$.

Proof. If A and B are well-ordered sets such that $\alpha = \circledbslash A$ and $\beta = \circledbslash B$, then $A \prec B$; that is, $A \approxeq S_x$ for some $x \in B$. Let $C = B - S_x$; C is well ordered and $C \neq \varnothing$, so if $\gamma = \circledbslash C$, then $\gamma > 0$. Now $B = S_x \oplus C$, $\alpha = \circledbslash S_x$ (because $S_x \approxeq A$), so $\beta = \alpha + \gamma$. For uniqueness, suppose $\beta = \alpha + \gamma = \alpha + \gamma'$, where, say, $\gamma < \gamma'$; that is, $\gamma' = \gamma + \delta (\delta > 0)$. Then

$$\beta = \alpha + \gamma' = \alpha + \gamma + \delta = \beta + \delta \ (\delta > 0),$$

which is in contradiction with the result of Theorem 9.14(i). Thus $\gamma = \gamma'$. ■

9.16 Theorem For any ordinal numbers α, β, γ, the following rules hold:

$$\begin{array}{rlrl}
\text{i)} & \alpha < \beta \Rightarrow \gamma + \alpha < \gamma + \beta, & \text{ii)} & \gamma + \alpha < \gamma + \beta \Rightarrow \alpha < \beta, \\
\text{iii)} & \alpha \leqslant \beta \Rightarrow \alpha + \gamma \leqslant \beta + \gamma, & \text{iv)} & \alpha + \gamma < \beta + \gamma \Rightarrow \alpha < \beta, \\
\text{v)} & \alpha < \beta, \gamma > 0 \Rightarrow \gamma\alpha < \gamma\beta, & \text{vi)} & \gamma\alpha < \gamma\beta \Rightarrow \alpha < \beta, \\
\text{vii)} & \alpha \leqslant \beta \Rightarrow \alpha\gamma \leqslant \beta\gamma, & \text{viii)} & \alpha\gamma < \beta\gamma \Rightarrow \alpha < \beta.
\end{array}$$

Proof. In (i), (iii), (v), and (vii) we assume that $\alpha < \beta$, hence we assume that there exists $\delta > 0$ such that $\beta = \alpha + \delta$. [*Note* : In (iii) and (vii), the case $\alpha = \beta$ is easily disposed of; indeed, if $\alpha = \beta$, then $\alpha + \gamma = \beta + \gamma$ and $\alpha\gamma = \beta\gamma$ (see Exercise 1, Exercise Set 9.2).]

i) $\gamma + \beta = \gamma + (\alpha + \delta) = (\gamma + \alpha) + \delta > \gamma + \alpha$ [this last relation is a consequence of 9.14(i)], so $\gamma + \beta > \gamma + \alpha$.

ii) Suppose $\gamma + \alpha < \gamma + \beta$. If $\alpha = \beta$, then $\gamma + \alpha = \gamma + \beta$ (Exercise 1, Exercise Set 9.2). If $\beta < \alpha$, then $\gamma + \beta < \gamma + \alpha$ by (i). Hence $\alpha < \beta$.

iii) Suppose, on the contrary, that $\beta + \gamma < \alpha + \gamma$; that is, $\alpha + (\delta + \gamma) < \alpha + \gamma$. Then $\delta + \gamma < \gamma$ by (ii), and this is impossible by 9.14(ii); thus $\alpha + \gamma \leqslant \beta + \gamma$.

iv) Suppose $\alpha + \gamma < \beta + \gamma$. If $\alpha = \beta$, then $\alpha + \gamma = \beta + \gamma$. If $\beta < \alpha$, then $\beta + \gamma \leqslant \alpha + \gamma$ by (iii). Thus $\alpha < \beta$.

v) $\gamma\beta = \gamma(\alpha + \delta) = \gamma\alpha + \gamma\delta > \gamma\alpha$ [this last relation holds by 9.14(i)].

vi) Suppose $\gamma\alpha < \gamma\beta$. If $\alpha = \beta$, then $\gamma\alpha = \gamma\beta$. If $\beta < \alpha$, then $\gamma\beta < \gamma\alpha$ by (v). Thus $\alpha < \beta$.

vii) We must show that $\alpha\gamma \leqslant (\alpha + \delta)\gamma$. Let $\alpha = \obslash A$, $\gamma = \obslash C$, $\delta = \obslash D$; $A \subseteq A \oplus D$,so

$$A \odot C \subseteq (A \oplus D) \odot C.$$

It follows by 4.63 that

$$A \odot C \mathrel{-\!\!\!\preccurlyeq} (A \oplus D) \odot C,$$

or $\alpha\gamma \leqslant (\alpha + \delta)\gamma$

viii) Suppose $\alpha\gamma < \beta\gamma$. If $\alpha = \beta$, then $\alpha\gamma = \beta\gamma$. If $\beta < \alpha$, then $\beta\gamma \leqslant \alpha\gamma$ by (vii). Hence $\alpha < \beta$. ■

9.17 Theorem

i) If $\gamma + \alpha = \gamma + \beta$, then $\alpha = \beta$.

ii) Assume $\gamma > 0$. If $\gamma\alpha = \gamma\beta$, then $\alpha = \beta$.

The proof, an immediate consequence of 9.16(i) and (v), is left as an exercise for the reader.

9.18 Lemma If $\gamma < \beta\alpha$, then there exist ordinals δ and ε such that $\gamma = \beta\delta + \varepsilon$, $\delta < \alpha$, and $\varepsilon < \beta$.

Proof. Let A, B, and C be well-ordered sets such that $\alpha = \obslash A$, $\beta = \obslash B$ and $\gamma = \obslash C$. Our assumption is that $C \mathrel{-\!\!\!<} B \odot A$, that is, $C \approxeq S_{(b,a)}$ for some $(b, a) \in B \odot A$.

Let $E = \{(x, a) : x < b\}$, that is, $E = S_b \odot \{a\}$; clearly $E \approxeq S_b$. We will show that

$$S_{(b,a)} = (B \odot S_a) \oplus E$$

(this relation is illustrated in Fig. 9.2).

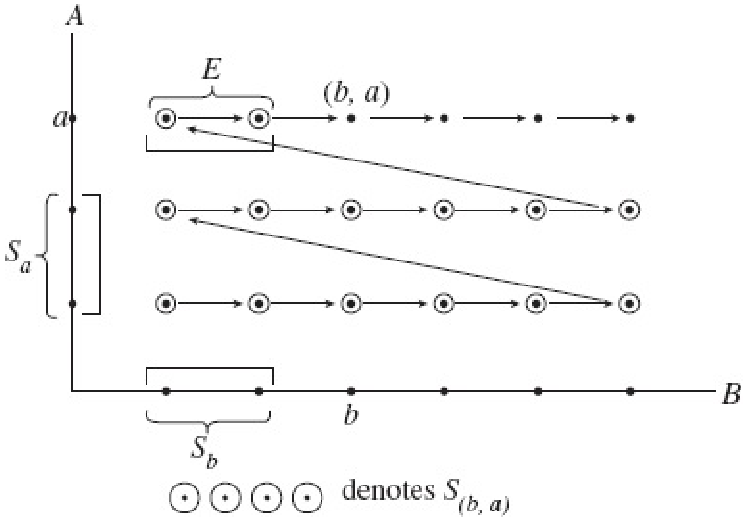


Fig. 9.2

Let $x \in B$, $y \in A$. Then

$$\begin{aligned}(x, y) \in S_{(b,a)} &\Leftrightarrow (x, y) < (b, a)\\ &\Leftrightarrow y < a \text{ (that is, } y \in S_a\text{) or } [y = a \text{ and } x < b]\\ &\Leftrightarrow (x, y) \in B \times S_a \text{ or } (x, y) \in E\\ &\Leftrightarrow (x, y) \in (B \times S_a) \cup E\\ &\text{Thus } S_{(b,a)} = (B \times S_a) \cup E.\end{aligned}$$

Now it is easy to verify that the ordering of $(B \odot S_a) \oplus E$ is the same as the ordering of $S_{(b,a)}$; the details are left as an exercise for the reader. We conclude that

$$S_{(b,a)} = (B \odot S_a) \oplus E.$$

Let $\varepsilon = ⦸E = ⦸S_b$ *and* $\delta = ⦸S_a$. By the definition of the relation $<$, $\varepsilon < \beta$ and $\delta < \alpha$. Now $\gamma = ⦸C = ⦸S_{(b, a)}$; thus $\gamma = \beta\delta + \varepsilon$. ■

It is very useful to note that the “division algorithm” for the natural numbers can be generalized to all the ordinal numbers.

9.19 Theorem If α and $\beta > 0$ are ordinals, then there exist unique ordinals ξ and ρ such that $\alpha = \beta\xi + \rho$ and $\rho < \beta$.

Proof

Existence. Since $\beta \geqslant 1$, we have $\beta\alpha \geqslant \alpha$. If $\beta\alpha = \alpha$, we are done; otherwise, $\alpha < \beta\alpha$, so by Lemma 9.18 there exist ordinals $\delta < \alpha$ and $\varepsilon < \beta$ such that $\alpha = \beta\delta + \varepsilon$, hence again we are done.

Uniqueness. Assume $\alpha = \beta\xi + \rho = \beta\xi' + \rho'$, where (say) $\xi' < \xi$; that is, $\xi = \xi' + \mu$ ($\mu > 0$). Then

$$\beta\xi' + \rho' = \beta\xi + \rho = \beta(\xi' + \mu) + \rho = \beta\xi' + \beta\mu + \rho,$$

so $\rho' = \beta\mu + \rho > \beta\mu \geqslant \beta$, which contradicts our assumption that $\rho' < \beta$. Thus we cannot have $\xi' < \xi$; by symmetry, we cannot have $\xi < \xi'$; thus $\xi = \xi'$. It follows by 9.17(i) that $\rho = \rho'$. ■

If α is an ordinal number, it is easy to see that $\alpha + 1$ is the immediate successor of α. Now let β be a non-zero ordinal number; if β has no immediate predecessor—that is, if β is not equal to $\alpha + 1$ for any ordinal α —then β is called a *limit ordinal*. Otherwise—that is, if β has an immediate predecessor—then β is called a *nonlimit ordinal*. Limit ordinals have the following useful properties.

9.20 Theorem

i) If α is a limit ordinal, there exists a unique ordinal ξ such that $\alpha = \omega\xi$.

ii) If α is a nonlimit ordinal, there exists a unique ordinal ξ and a unique finite ordinal $n \neq 0$ such that $\alpha = \omega\xi + n$.

Proof. We will prove the existence assertions; the uniqueness assertions' are left as an exercise for the reader.

i) By Theorem 9.19, there exist unique ordinals ξ and ρ such that $\alpha = \omega\xi + \rho$ and $\rho < \omega$. But if $\rho < \omega$, then ρ must be finite. But then ρ must be 0; for otherwise $\rho = m + 1$ for some finite m, hence $\alpha = \omega\xi + m + 1$ would not be a limit ordinal.

ii) As above, $\alpha = \omega\xi + \rho$, where ρ is finite. Now $\rho \neq 0$; for if $\rho = 0$, then $\alpha = \omega\xi$, which is impossible because $\omega\xi$ is a limit ordinal (see Exercise 6, Exercise Set 9.3). ■

EXERCISES 9.3

1. Prove that $1 + \alpha = \alpha$ if an only if $\alpha \geqslant \omega$.
2. An ordinal number $\rho > 0$ is called *irreducible* if there exists no pair of ordinals α, β such that $\alpha < \rho$, $\beta < \rho$, and $\alpha + \beta = \rho$. Prove the following:

   a) An ordinal ρ is irreducible if and only if $\pi + \rho = \rho$ for every ordinal $\pi < \rho$.

   b) Suppose $\rho > 1$ and $\varepsilon > 0$; $\varepsilon\rho$ irreducible $\Rightarrow \rho$ irreducible.

   c) If ρ is irreducible and $0 < \mu < \rho$, then there exists an irreducible ordinal ξ such that $\rho = \mu\xi$.

   d) Suppose $\alpha > 0$; the set of all irreducible ordinals $\leqslant \alpha$ has a greatest element. [*Hint*: Consider the set of all β such that $\alpha = \rho + \beta$ for some irreducible ρ.]
3. Show that an ordinal α is a limit ordinal if and only if

$$\beta < \alpha \Rightarrow (\beta + 1) < \alpha.$$

4. Let γ be a *non*limit ordinal. Prove the following.

   a) $\alpha < \beta \Rightarrow \alpha\gamma < \beta\gamma$, b) $\alpha\gamma = \beta\gamma \Rightarrow \alpha = \beta$.
5. Let $\beta \neq 0$. Prove that $\alpha + \beta$ is a limit ordinal if and only if β is a limit ordinal.
6. Let $\alpha, \beta \neq 0$. Prove that $\alpha\beta$ is a limit ordinal if and only if α is a limit ordinal or β is a limit ordinal. [*Hint*: Use Exercise 3 and Lemma 9.18.]
7. Prove that $n\omega = \omega$, $\forall n \in \omega$. [*Hint*: Use Theorem 9.19 to “divide” ω by n.]
8. Let $\alpha \neq 0$. Prove that α is a limit ordinal if and only if $n\alpha = \alpha$, for every finite n. [*Hint*: Use Exercise 7 and Theorem 9.20(i). For the converse, use Exercise 4(b).]
9. a) Use induction to prove that if γ is an infinite ordinal, then $(\gamma + 1)n = \gamma n + 1$ for all finite n. [Use Exercise 1.]

   b) Prove that $\forall\gamma > 0$, $(\gamma + 1)\omega = \gamma\omega$. [*Hint*: If γ is infinite, assume $\gamma\omega < (\gamma + 1)\omega$ and use Theorem 9.18 to arrive at a contradiction. If γ is finite, use Exercise 7.]

   c) Conclude that if β is any limit ordinal, then

$$(\gamma + 1)\beta = \gamma\beta, \quad \forall\gamma > 0.$$

10. Let α be a *non*limit ordinal. Prove that $\forall\gamma > 0$, $(\gamma + 1)\alpha > \gamma\,\alpha$. [*Hint*: Use Exercise 4.]
11. If α is an infinite ordinal and $\beta \neq 0$ is a *non*limit ordinal, prove that

$$(\alpha + 1)\beta = \alpha\beta + 1.$$

[*Hint*: Use Theorem 9.20(ii) and assume $\alpha\beta + 1 < (\alpha + 1)\beta$ to arrive at a contradiction. Use Exercise 9.]

12. a) If α is a limit ordinal, prove that $\alpha = \sup\{\beta : \beta < \alpha\}$. [Use Exercise 3.]

b) If α is a limit ordinal and β is any ordinal, prove that

$$\beta + \alpha = \sup\{\beta + \mu : \mu < \alpha\}.$$

[*Hint*: If γ is an upper bound of $\{\beta + \mu : \mu < \alpha\}$, then $\gamma > \beta$, that is, $\gamma = \beta + \delta$; δ proves to be an upper bound of $\{\mu : \mu < \alpha\}$.]

13. If α is a limit ordinal and β is any ordinal, prove that

$$\beta\alpha = \sup\{\beta\mu : \mu < \alpha\}.$$

[*Hint*: If γ is an upper bound of $\{\beta\mu : \mu < \alpha\}$, then $\gamma = \alpha\delta + \rho(\rho < \alpha)$. Note that $\mu < \alpha \Rightarrow \gamma \geqslant \alpha(\mu + 1)$and conclude that δ is an upper bound of $\{\mu : \mu < \alpha\}$.]

14. Prove Theorem 9.17.

15. Prove the uniqueness assertions of Theorem 9.20.

4 THE ALEPHS AND THE CONTINUUM HYPOTHESIS

In Chapter 8 it was proven that the relation $\leqslant$ among cardinals is a well-ordering; hence there is a smallest infinite cardinal, a next greater infinite cardinal, and so on; every infinite cardinal has a uniquely determined immediate successor. It follows that the infinite cardinal numbers can be ranked in “first, second, third, …” order. This opportunity of ranking the cardinals—and using the ordinals to designate the ranks—has valuable applications in mathematics.

We will proceed to show that there is an isomorphism between the class of all the infinite cardinals and the class of all the ordinals.

9.21 Theorem Let IC be the class of all the infinite cardinals and let OR be the class of all the ordinals. There exists an isomorphism from IC to OR.

Proof We begin by noting that every initial segment of IC is a set. If a is an infinite cardinal, then $\{b \in IC : b < a\}$ is an initial segment of IC denoted here by I_a. We well-order a, and for each $x \in a$, $S_x = \{u \in a : u < x\}$ is an initial segment of a. Let $\phi : a \to I_a$ be defined by $\phi(x) = \#S_x$ for each $x \in a$. ϕ is surjective, because if $b \in I_a$, (hence $b < a$) then b is isomorphic (hence equipotent) with some initial segment of a. (The alternative, namely a isomorphic with some initial segment of b, contradicts $b < a$). Thus ϕ is surjective. Since a is a set, I_a is a set by Axiom A9. Likewise, every initial segment of OR is a set.

Now IC and OR are both well-ordered classes, hence by 4.62, exactly one of the following three cases must hold: (a) IC $\approxeq$ OR, (b) IC is similar to an initial segment of OR, (c) OR is similar to an initial segment of IC. Suppose for a moment that (c) holds. From the previous paragraph, every initial segment of IC is a set, hence by 2.36, OR is a set. But we have proved that OR is a proper class, hence (c) cannot hold. Analogously, (b) cannot hold, which proves that (a) holds. ■

If a is an infinite cardinal number, the ordinal $\Omega(a)$ is called the *ordinal rank* of a. Note that Ω is an isomorphism; thus, if a is the least infinite cardinal, then $\Omega(a)= 0$; if a is the next greater infinite cardinal, then $\Omega(a)= 1$, and so on.

The infinite cardinals are often called *alephs*. If a is an infinite cardinal and $\alpha = \Omega(a)$, we frequently write

$$a = \aleph_\alpha,$$

Thus the first few infinite cardinals are $\aleph_{0_0}$, $\aleph_{0_1}$, $\aleph_{0_2}$, …

Theorem 9.21 has the following simple consequences:

9.22

i) $\aleph_{0_\alpha} = \aleph_{0_\beta} \Rightarrow \alpha = \beta$,

ii) $\alpha = \beta \Rightarrow \aleph_{0_\alpha} = \aleph_{0_\beta}$,

iii) $\aleph_{0_\alpha} < \aleph_{0_\beta} \Rightarrow \alpha < \beta$,

iv) $\alpha < \beta \Rightarrow \aleph_{0_\alpha} < \aleph_{0_\beta}$

We have seen that every infinite cardinal number a has an immediate successor—but what exactly *is* the immediate successor of a? We know that 2^a is greater than a, but is there any cardinal number between a and 2^a? Let us ask a more specific question: We have seen that $\aleph_{0_0}$ is the cardinal number of denumerable sets (for the cardinal number of ω is the least infinite cardinal), and that $2^{\aleph_{00}}$ is the cardinal number of the real numbers; is there a cardinal number between these two?

The early set theorists proposed the hypothesis that there is no cardinal between $\aleph_{0_0}$ and $2^{\aleph_{00}}$, and named it the *continuum hypothesis*—for it is equivalent to saying that every set of real numbers which is not denumerable has the power of the real numbers, called the “power of the continuum.” An obvious extension of this conjecture is the statement: For every infinite cardinal number a, there is no cardinal between a and 2^a; this is called the *generalized continuum hypothesis*.

Continuum Hypothesis. There does not exist any cardinal c such that $\aleph_{0_0} < c < 2^{\aleph_{00}}$

Generalized Continuum Hypothesis. If a is any infinite cardinal, there does not exist any cardinal c such that $a < c < 2^a$.

It has been proven in recent years that the continuum hypothesis and the generalized continuum hypothesis cannot be proven from the other axioms of set theory, and do not contradict these. Hence their status is analogous to that of Euclid’s “Fifth Postulate” in geometry. We may postulate them or deny them, in each case getting a consistent theory of cardinal numbers.

EXERCISES 9.4

1. Prove that the generalized continuum hypothesis is equivalent to

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

2. Assuming the generalized continuum hypothesis, prove the following:

$$\text{If } \alpha \leqslant \beta, \quad \text{then} \quad \aleph_\alpha^{\aleph_\beta} = \aleph_{\beta+1}.$$

3. Assuming the generalized continuum hypothesis, prove the following:

$$\text{For arbitrary cardinals } a, b, \quad a < b \Rightarrow 2^a < 2^b.$$

Further problems on the alephs are given in Exercise Set 9.5.

5 CONSTRUCTION OF THE ORDINALS AND CARDINALS

We said, in the introduction to this chapter, that it is possible to construct sets which satisfy Conditions 01 and 02 of the Axiom of Ordinality. The chief purpose of this construction is to prove that we can dispense with the Axiom of Ordinality by actually producing the sets whose existence the axiom asserts.

Our process of construction is based upon the same idea—outlined on page 125—that we used to construct the natural numbers. We begin by defining

$$\begin{aligned} 0 &= \emptyset, \\ 1 &= \{\emptyset\}, \\ 2 &= \{\emptyset, \{\emptyset\}\}, \quad \text{etc.} \end{aligned}$$

If A is a set, we define the *successor of A* to be the set A^+, given by

$$A^+ = A \cup \{A\}.$$

Thus $0 = \emptyset$, $1 = 0^+$, $2 = 1^+$, and so on. This time, however, we will go further than we did in Chapter 7. Starting with ω, we define

$$\begin{aligned} \omega + 1 &= \omega^+, \\ \omega + 2 &= (\omega + 1)^+, \quad \text{etc.} \end{aligned}$$

Then, starting with $\omega + \omega = \omega 2$, we get

$$\omega 2 + 1, \omega 2 + 2, \ldots, \omega 3, \omega 3 + 1, \ldots, \omega 4, \ldots, \ldots, \omega\omega,$$

and so on.

This is the *basic idea* of our construction process, but we will not proceed exactly in this fashion. Instead of starting with 0 and constructing successive sets one by one, we will define all the sets simultaneously. This can be accomplished in the following way.

The “elementhood” relation ∈ is not, generally speaking, an order relation; for example, if $x \in A$ and $A \in B$, it does not necessarily follow that $x \in B$. However, there are special cases where ∈ does behave as if it were an order relation; one of these cases concerns us here.

9.23 Definition A class A is said to be ∈-*ordered* if it is ordered by the relation ∈. It is understood here that the relation ∈ is a strict order relation <. Consequently, to say that A is ∈-ordered is to say it has the following properties $\forall a, b, c \in A$:

$a \in a$.

$a \in b \Rightarrow b \in a$.

$(a \in b) \wedge (b \in c) \Rightarrow a \in c$

9.24 *Remark*. It is immediate that the transitive law (c) holds iff $b \in c \Rightarrow b \subseteq c$.

By Definition 6.5, a set A is said to be *transitive* if: $(x \in A) \wedge (y \in x) \Rightarrow y \in A$. Note that this property is not the same as saying that (c) above holds in A. Property (c) tells us that *every element of A* is transitive, but not that A itself is transitive. It is immediate that A is transitive iff $x \in A \Rightarrow x \subseteq A$. Here is our "abstract" definition of ordinals:

9.25 Definition A set A is called an *ordinal* if A is transitive and ordered by $\in$. This means that $\in$ satisfies the three conditions of Definition 9.23 on A, and A is a transitive set by Definition 6.5.

9.26 Lemma Let α be an ordinal.

i) Every element of an ordinal is an ordinal.

ii) If $x \in \alpha$, then $x = S_x$, where S_x is the initial segment $\{y \in \alpha : y \in x\}$.

Proof

i) Suppose $x \in \alpha$; show that x is an ordinal. Because α is transitive, $x \subseteq \alpha$. So if a, b, c are elements of x, they are also elements of α, hence satisfy conditions (a), (b), (c) of 9.23. Likewise, if u, v, w are elements of x, they are elements of α, hence $u \in v \in x \Rightarrow u \in x$.

ii) If $y \in x$ then (because A is a transitive set) $y \in \alpha$. Thus $y \in S_x$. Conversely, if $y \in S_x$ then by the definition of S_x, $y \in x$. Thus, $x = S_x$. ■

From Lemma 9.26 we draw two important conclusions: (1) Every ordinal is a set of ordinals, and (2) Every ordinal α is the set of all ordinals $\beta < \alpha$.

We let the symbol **OR** stand for the class of all the ordinals. Since ordinals are sets, $\in$ is a relation between them, and our first objective is to show that **OR** satisfies 9.23(a), (b) and (c). We aim also to show that any two ordinals α and β are comparable by the order relation $\in$. In fact, let's begin with that:

9.27 Theorem Let α and β be ordinals. If $\alpha \neq \beta$, then either $\alpha \in \beta$ or $\beta \in \alpha$.

Proof. We shall call two sets x and y *incomparable* if $x \neq y$, $x \notin y$ and $y \notin x$. Let A be the following subset of **OR**: $A = \{x \in \mathbf{OR} : \exists y \in \mathbf{OR}$ such that x and y are incomparable$\}$. We will show that A is empty. We reason by contradiction and assume A is nonempty. At this point we shall make our first use of the Axiom of Foundation (A8), which states that every nonempty set A contains an element disjoint from A. That is: $\exists a \in A$ such that $a \cap A = \emptyset$. In that case, the set $B = \{y \in \mathbf{OR}$: y is incomparable with $a\}$ is not empty. As above, by (A8) there is an element $b \in B$ such that $b \cap B = \emptyset$. It will now be shown that $a \subseteq b$.

If $z \in a$ then z is an ordinal, and $z \notin A$ because $a \cap A = \emptyset$. For every $y \in \mathbf{OR}$, y and z are comparable. In particular, b and z are comparable. Now if it were the case that $z = b$ that would imply $b \in a$ hence $b \notin B$, which is contradiction. Similarly, if it were the case that $b \in z$, then $b \in a$ because a is transitive, hence $b \notin B$. Again a contradiction. Thus, $z \in b$, and since z was an arbitrary element of a, this proves that $a \subseteq b$. In the same fashion you can show $b \subseteq A$, hence $a = b$. This contradiction proves the theorem. ■

9.28 Lemma If α and β are ordinals then $\alpha \in \beta \Leftrightarrow \alpha \subset \beta$. ($\alpha \subset \beta$ means $\alpha \subseteq \beta$ but $\alpha = \beta$.)

Proof. Half of this claim is true from Remark 9.24 with 9.23(a). Now suppose $\alpha \subset \beta$: It must be shown that $\alpha \in \beta$. By Theorem 9.27, either $\alpha \in \beta$ or $\beta \in \alpha$. The latter is impossible, because from $\alpha \subseteq \beta$ we would get $\beta \in \beta$, contrary to 9.23(a). Thus $\alpha \in \beta$. ■

The class of all the ordinals is denoted by **OR**. The relation $\in$ is a well-defined relation on **OR**, and we wish to show that $\in$ is an order relation on **OR**. In order to do this, we must show that $\in$ satisfies Properties 9.23(a), (b) and (c) on **OR**. We also show that **OR** is a transitive class.

9.29 Theorem The class **OR** of all the ordinals is ordered by $\in$.

Proof. Let $\alpha, \beta, \gamma \in$ **OR**:

If $\alpha \in \alpha$ then α is an element of an ordinal, hence satisfies 9.23(a). Thus, $\alpha \notin \alpha$.
We show the transitive law first: Suppose $\alpha \in \beta$ and $\beta \in \gamma$: We make three uses of Lemma 9.28: $\alpha \subset \beta$ and $\beta \subset \gamma$ hence $\alpha \subset \gamma$ so $\alpha \in \gamma$.
Suppose $\alpha \in \beta$. If $\beta \in \alpha$ then (because every ordinal is transitive) $\alpha \in \alpha$, which is false from (a). ■

9.30 Lemma If α is an ordinal then $\alpha^{+} = \alpha \cup \{\alpha\}$ is an ordinal.
The proof of this lemma is a simple verification, and is left as an exercise.

9.31 Theorem

a) If α is an ordinal then $\alpha = S_{\alpha}$.

b) **OR** is a transitive class.

Proof

a) It must be shown that if α is an ordinal then $\alpha = S_{\alpha}$. From 9.26(ii), all we need is to show that α is an element of an ordinal. It is, because $\alpha \in \alpha^{+}$.

b) Prove that $x \in$ **OR** $\Rightarrow x \subset$ **OR**. In other words, if x is an ordinal, then x is a set of ordinals. This follows from Part (a). ■

9.32 Theorem Every non-empty class A of ordinals has a least element.

Proof It will be shown that if A is any class of ordinals, then $\bigcap A$ is an ordinal, is an element of A, and is therefore the least ordinal in A. If $x \in \bigcap A$ then for any $y \in A$, $x \in y$. So now, if $z \in x$ then $z \in y$ because ordinals are transitive. Since this is true for each $y \in A$, it follows that $z \in \bigcap A$. This proves that $\bigcap A$ is transitive. It is clear that $\bigcap A$ is ordered by $\in$, because each ordinal in A satisfies 9.23(a), (b) and (c), hence so does their intersection. Thus, $\bigcap A$ is an ordinal.

Finally, either $\bigcap A = A$ (which implies A is a singleton $\{\alpha\}$ so $\alpha \in A$ is the least element of A) or $\bigcap A \subset A$. Then from 9.28, $\bigcap A \in A$. It is immediate that if $\gamma \in A$ then $\bigcap A \subset \gamma$, so $\bigcap A \in \gamma$. Thus $\bigcap A$ is the least element of A. ■

This theorem is very important, because it shows that the class **OR** of all ordinals is well-ordered by the relation $\in$. Moreover, since from 9.26(i) every ordinal α is a subset of **OR**, this shows that every ordinal is well-ordered by $\in$. One essential fact about **OR** is this:

9.33 Theorem OR is a proper class.

Proof. We have shown that **OR** has all the properties of an ordinal. Thus if **OR** were a set, it would be an ordinal, and then we would have **OR** $\in$ **OR**, which is impossible by 9.23(c). ■

Finally, we come to the most important property of **OR**—the reason for inventing ordinals in the first place:

9.34 Theorem Every well-ordered set A is isomorphic with a unique ordinal.

Proof Since A and **OR** are well-ordered classes, it follows from Theorem 4.62 that either A is isomorphic with an initial segment of **OR**, or **OR** is isomorphic with an initial segment of A, or **OR** is isomorphic with A. The last two are impossible, because from Axiom A9, this would make **OR** a set, whereas **OR** is a proper class. Thus, A is isomorphic to the initial segment $S_\alpha = \alpha$ of **OR**. The proof of uniqueness is left as an exercise. ■

9.35 *Remark*. Ordinal numbers have many applications in mathematics. Here is a brief summary of the high points that it is important to retain: Every ordinal α is an $\in$-well-ordered set equal to the set of all ordinals preceding it: $\alpha = \{\beta \in \mathbf{OR} : \beta < \alpha\}$. Moreover, $\alpha \in \beta \Leftrightarrow \alpha \subset \beta$. Thus, when we think of ordinals as sets (which they are), each ordinal is a chain of sets: α is the chain of all the sets $\beta \subset \alpha$. As a set, $\alpha = \bigcup\{\beta \in \mathbf{OR} : \beta \subset \alpha\}$. In fact, if A is any class of ordinals, then A may correctly be viewed as a chain of sets. Consequently, any bounded class A of ordinals has a least upper bound sup(A) = $\bigcup A$.

9.36 *Remark*. Let $C \subseteq \mathbf{OR}$ be a class of ordinal numbers, and suppose the following three conditions hold: (a) $0 \in C$. (b) If $\alpha \in C$ then $\alpha^+ \in C$. (c) Let α be a limit ordinal: If $\beta \in C$ for all $\beta < \alpha$ then $\alpha \in C$. Then $C = \mathbf{OR}$. This fact is easy to prove: Indeed, if $C \neq \mathbf{OR}$, let D be the class of all ordinals γ such that $\gamma \notin C$. By assumption, $D \neq \emptyset$, so from 9.32, D has a least element μ. It is easy now to get a contradiction.

If A is an arbitrary set, consider the class of all the ordinals equipotent with A; this class has a unique least element, which we call the *initial ordinal* equipotent with A. It is trivial, now, to verify that the class of all the initial ordinals satisfies Conditions K1 and K2 of the Axiom of Cardinality. Thus we are justified in making the following definition.

9.37 Definition By a *cardinal number* we mean an initial ordinal.

Thus, the class CD of the cardinal numbers is the class of all the initial ordinals.

We have thus fulfilled our promise of actually producing sets to serve as the cardinal numbers and the ordinal numbers.

We noted earlier that every natural number is a transitive, ∈-well-ordered set, that is, an ordinal number. It is immediate, too, that every natural number is an *initial* ordinal number. Thus, in our construction, the natural numbers coincide with the finite ordinals, as well as with the finite cardinals.

The reader should note that everything we have proved in Chapters 8 and 9 about the class CD of the cardinal numbers has depended solely upon Conditions K1 and K2 of the Axiom of Cardinality. Thus everything we have already said about the cardinals holds, without any alteration, for the class CD defined by 9.37. In particular, 9.22 still holds, that is,

$$\alpha \leftrightarrow \aleph_\alpha$$

is an isomorphism between OR and the class of all the infinite cardinals.

It is important to note that by 9.37, $\aleph_{0\alpha}$ is both a cardinal and an ordinal (specifically, an initial ordinal). In order to avoid any confusion, it is common practice in mathematics to write

$$\aleph_\alpha = \omega_\alpha$$

for every infinite cardinal $\aleph_{0\alpha}$, and to treat $\aleph_{0\alpha}$ as a cardinal number and ω_α as an ordinal number. In other words, the number in question is denoted by $\aleph_{0\alpha}$ when it is used as a cardinal, and by ω_α when it is used as an ordinal. For example, $\aleph_{0\alpha} + \aleph_{0\beta}$ designates the cardinal sum of the two numbers, whereas $\omega_\alpha + \omega_\rho$ designates their ordinal sum.

EXERCISES 9.5

1. For each ordinal number α, prove that $\alpha \leqslant \omega_\alpha$; conclude that $\#\alpha \leqslant \aleph_{0\alpha}$. [*Hint*: Use 4.58.]
2. Prove that if β is a limit ordinal, then $\aleph_{0\beta} = \sup\{\aleph_{0\gamma} : \gamma < \beta\}$. [*Hint*: Use 9.22.]
3. Prove that $\sum_{\gamma \leqslant \mu} \aleph_\gamma = \aleph_\mu$.
4. If μ is a limit ordinal, prove that $\sum_{\gamma \leqslant \mu} \aleph_\gamma = \aleph_\mu$.

10

Transfinite Recursion. Selected Topics in the Theory of Ordinals and Cardinals

1 TRANSFINITE RECURSION

In Chapter 6 we discussed the notions of finite induction and finite recursion; finite induction is method of *proof by induction* and is familiar to most students of elementary algebra. Finite recursion is a method of *definition by induction*. Proof by induction works as follows: A theorem is first shown to be true for 0. Then it is shown that if the theorem is true for n, it must likewise be true for $n + 1$. We are then able to conclude that the theorem is true for all natural numbers. Induction rests on the same principle, but is a way of defining a function. First, we define the value of the function at 0. Then we use its value at n to define its value at $n + 1$. In this way, its value is defined for every natural number.

Our purpose here is to go beyond the natural numbers and use the same principle on the class of all the ordinals. Transfinite induction has been described briefly in Section 4.5. The chief difference between conventional induction and transfinite induction is that there are two kinds of induction step: One for the case where α is a successor ordinal, and one for the case where α is a limit ordinal.

The two cases must obviously be treated differently, because if α is a successor ordinal, say $\alpha = \beta^+$, then the task is to show that if a property is true for β then it is true for β^+. On the other hand, if α is a limit ordinal, the task is to show that if the property is true for all $\beta < \alpha$, then it is true also for α.

The difference between induction and recursion is clear: In induction the objective is to prove that a property $P(\alpha)$ is true for every ordinal α. In recursion the objective is to *assign a value* to a function $F(\alpha)$ for every ordinal α. The idea is the same, but the added complication is that the value you assign is in a set A which is the range of F.

Suppose α is a successor ordinal, $\alpha = \beta^+$. The idea, in recursion as in induction, is that the value of $F(\beta^+)$ is obtained directly from the value of $F(\beta)$—in other words, by performing some operation on $F(\beta)$. If G is the symbol assigned to that operation, then

$$F(\beta^+) = G(F(\beta))$$

On the other hand, if α is a limit ordinal, what you do to all the values $F(\beta)$, $\beta < \alpha$ in order to get $F(\alpha)$ is to assign the "next value" to $F(\alpha)$, which is their least upper bound:

$$F(\alpha) = \sup\{F(\beta) : \beta < \alpha\}$$

As we have just seen, G is an operation on the range of F. So the next question is: What is the range of F ? Let's use the symbol A for the range of F. Different choices of A give you different versions of the recursion theorem. You want to be as noncommittal as possible so as to allow the recursion theorem to be used in a wide range of applications. So, let's say that A is any subclass of the class V of all sets. Since anything in mathematics can be construed as a set, this is very satisfactory. Given any set of sets, their least upper bound is usually their union. Thus, our induction step for limit ordinals may be written thus:

$$F(\alpha) = \bigcup\{F(\beta) : \beta < \alpha\}$$

It is good to recall here, from 9.35, that **OR** is a chain of sets ordered by ⊂. Thus, any subclass $C \subseteq$ **OR** is likewise a chain of sets, so its union is its least upper bound. That is, sup $C = \bigcup C$. It is also good to recall that every ordinal α is equal to the initial segment S_α of **OR**, and since S_α is a chain of sets, $\alpha = \bigcup\{\beta : \beta < \alpha\}$. So here we have it:

Transfinite Recursion Theorem Let $A \subseteq V$ be any class of sets, let $G : A \to A$ be a function, and let $a \in A$. Then there exists a unique function F : **OR** → A such that

i) $F(0) = a$
ii) $F(\alpha^+) = G(F(\alpha))$
iii) $F(\alpha) = \bigcup\{F(\beta) : \beta < \alpha\}$ if α is a limit ordinal.

Proof. The proof will flow smoothly if we think of all our functions as sets of ordered pairs—which is actually what they are. Our task now is to show that for every ordinal α there is exactly one element x_α such that the ordered pair (α, x_α) is in F. In the mind's eye, we should think of F as being built up stage by stage—adding a new ordered pair at each stage—until F is "full".

Condition (i) of the theorem tells us that the pair $(0,a)$ is in F. Thus, F is not empty. If dom(F) = **OR**, we are done. If not, let α be the least ordinal such that no pair (α, x_α) is in F for any element x_α. (Use Theorem 9.32.) Consider two cases:

a) α is a successor ordinal, $\alpha = \beta^+$. By assumption, F contains an ordered pair (β, x_β) for some element x_β. Then from Condition (ii), the ordered pair $(\beta^+, G(x_\beta))$ is in F: This contradicts the assumption that no pair (α, x_α) is in F.

b) α is a limit ordinal. Then by assumption, for all $\beta < \alpha$, there are ordered pairs (β, x_β) in F. That is, $\{(\beta, x_\beta) : \beta < \alpha\} \subseteq F$. Then by the rule of induction (iii), there is $x_\alpha = \bigcup_{\beta<\alpha} x_\beta$ such that $(\alpha, x_\alpha) \in F$. This shows that for every α, some pair (α, x_α) is in F.

We use induction on α to prove that for each ordinal α, there is only one x_α such that $(\alpha, x_\alpha) \in F$. Suppose that for all $\beta < \alpha$, there is just one ordered pair $(\beta, x_\beta) \in F$. That is, if $(\beta, x_\beta) \in F$ and $(\beta, x'_\beta) \in F$ then $x_\beta = x'_\beta$. Let α be a successor ordinal, $\alpha = \beta^+$, and suppose there are different elements x_α and x'_β such that $(\alpha, x_\alpha) \in F$ and $(\alpha, x'_\alpha) \in F$. From Condition (ii), $x_\alpha = x_{\beta+} = G(x_\beta) = G(x'_\beta) = x'_{\beta+} = x'_\alpha$. Now let α be a limit ordinal, and suppose $(\alpha, x_\alpha) \in F$ and $(\alpha, x'_\alpha) \in F$. From (iii), $x_\alpha = \bigcup_{\beta<\alpha} x_\beta = \bigcup_{\beta<\alpha} x'_\beta = x'_\alpha$. This proves that the set F of ordered pairs is a function.

From 9.36, the domain of F is all of **OR**. It has thus been shown that F is a function whose domain is **OR** and range is in A, and that Conditions (i)–(iii) determine F uniquely. ■

Remark. The reasoning in Paragraph 3 of this proof may be interpreted as follows: For every ordinal α,if $F\restriction S_\alpha$ is the only function with domain S_α that satisfies Conditions (i)–(iii), then $F\restriction S_{\alpha+} = (F\restriction S_\alpha) \cup (\alpha, x_\alpha)$ is the only function with domain S_α+ that satisfies (i)–(iii). But $F = \bigcup_{\alpha\in OR}(F \restriction S_\alpha)$, so F is the unique function with domain **OR** that satisfies (i)–(iii).

Just as finite recursion was used in Chapter 6 to define the addition and multiplication of natural numbers, transfinite recursion may be used to give alternative definitions for the addition and multiplication of ordinal numbers.

10.2 Definition For any arbitrary $\alpha \in \mathbf{OR}$, we define a function $\sigma_\alpha : OR \to OR$ as follows:

a) $\sigma\alpha(0) = \alpha$,

b) $\sigma\alpha(\beta^+) = [\sigma\alpha(\beta)]^+$,

c) $\sigma\alpha(\beta) = \sup\{\sigma\alpha(\gamma) : \gamma < \beta\}$ if β is a limit ordinal.

Theorem 10.1 guarantees the existence of a unique function σ_α satisfying (a), (b), (c).

10.3 Theorem For arbitrary ordinals α, β, $\sigma_\alpha(\beta) = \alpha + \beta$.

Proof. If $\beta = 0$, then $\sigma_\alpha(0) = \alpha = \alpha + 0$. By induction, let us suppose now that the theorem holds for all $\gamma < \beta$. If β is a limit ordinal, then

$$\begin{aligned} \sigma_\alpha(\beta) &= \sup\{\sigma_\alpha(\gamma) : \gamma < \beta\} && \text{by 10.2(c)} \\ &= \sup\{\alpha + \gamma : \gamma < \beta\} && \text{by the hypothesis of induction} \\ &= \alpha + \beta && \text{by Exercise 12(b), Exercise Set 9.3.} \end{aligned}$$

If β is a nonlimit ordinal, $\beta = \delta^+$, then

$$\begin{aligned} \sigma_\alpha(\beta) = \sigma_\alpha(\delta^+) &= [\sigma_\alpha(\delta)]^+ && \text{by 10.2(b)} \\ &= (\alpha + \delta)^+ && \text{by the hypothesis of induction} \\ &= \alpha + \delta^+ && \text{immediate consequence of 9.9(i)} \\ &= \alpha + \beta. \ \blacksquare \end{aligned}$$

Theorem 10.3 tells us that if we define the addition of ordinal numbers by

10.4 $$\alpha + \beta = \sigma_\alpha(\beta), \quad \sigma_\alpha \text{ given by 10.2,}$$

then 10.4 is equivalent to Definition 9.8.

10.4 may also be written in the following form:

10.5 a) $\alpha + 0 = \alpha$,

b) $\alpha + \beta^+ = (\alpha + \beta)^+$,

c) $\alpha + \beta = \sup\{\alpha + \gamma : \gamma < \beta\}$ if β is a limit ordinal.

Since 10.5 is equivalent to Definition 9.8, we will henceforth consider 10.5 to define the addition of ordinal numbers.

10.6 Definition For an arbitrary $\alpha \in$ OR, we define a function π_α : OR → OR as follows:

a) $\pi_\alpha(0) = 0$,

b) $\pi_\alpha(\beta^+) = \pi_\alpha(\beta) + \alpha$,

c) $\pi_\alpha(\beta) = \sup\{\pi_\alpha(\gamma) : \gamma < \beta\}$ if β is a limit ordinal.

Theorem 10.1 guarantees the existence of a function π_α satisfying the conditions (a), (b), and (c).

10.7 Theorem For arbitrary ordinals α, β, $\pi_\alpha(\beta) = \alpha\beta$.

Proof. If $\beta = 0$, then $\pi_\alpha(0) = 0 = \alpha 0$. By induction, let us suppose that the theorem holds for every $\gamma < \beta$; that is, $\pi_\alpha(\gamma) = \alpha\gamma$, $\forall\gamma < \beta$. If β is a limit ordinal then

$$\begin{aligned} \pi_\alpha(\beta) &= \sup\{\pi_\alpha(\gamma) : \gamma < \beta\} && \text{by 10.6(c)} \\ &= \sup\{\alpha\gamma : \gamma < \beta\} && \text{by the hypothesis of induction} \\ &= \alpha\beta && \text{by Exercise 13, Exercise Set 9.3.} \end{aligned}$$

If β is a nonlimit ordinal, $\beta = \delta^+$, then

$$\begin{aligned} \pi_\alpha(\beta) = \pi_\alpha(\delta^+) &= \pi_\alpha(\delta) + \alpha && \text{by 10.6(b)} \\ &= \alpha\delta + \alpha && \text{by the hypothesis of induction} \\ &= \alpha(\delta + 1) && \text{by 9.9(iii)} \\ &= \alpha\beta. \ \blacksquare \end{aligned}$$

Theorem 10.7 tells us that if we define the multiplication of ordinal numbers by

10.8 $$\alpha\beta = \pi_\alpha(\beta), \quad \text{where } \pi_\alpha \text{ is given by 10.6,}$$

then 10.8 is equivalent to Definition 9.8.

10.8 may also be written in the following form:

10.9 a) $\alpha 0 = 0$,

b) $\alpha\beta^+ = \alpha\beta + \alpha$,

c) $\alpha\beta = \sup\{\alpha\gamma : \gamma < \beta\}$ if β is a limit ordinal.

Since 10.9 is equivalent to 9.8, we will henceforth consider 10.9 to be the definition of ordinal multiplication.

2 PROPERTIES OF ORDINAL EXPONENTIATION

10.5 and 10.9 provide us with an alternative way of defining the addition and multiplication of ordinal numbers, using transfinite recursion. We will use this new (and in many ways, more convenient) approach to define ordinal exponentiation.

10.10 Definition If $\alpha \neq 0$ is any ordinal, we define a function η_α : OR → OR as follows:

a) $\eta_\alpha(0) = 1$.

b) $\eta_\alpha(\beta^+) = \eta_\alpha(\beta)\alpha$,

c) $\eta_\alpha(\beta) = \sup\{\eta_\alpha(\gamma) : \gamma < \beta\}$ if β is a limit ordinal.

Theorem 10.1 guarantees the existence of a function η_α : OR → OR which satisfies conditions (a), (b), and (c) above.

10.11 Definition We define ordinal exponentiation as follows: If α and β are arbitrary ordinals, we let

$$\alpha^\beta = \eta_\alpha(\beta) \quad \text{if} \quad \alpha \neq 0,$$

and $0^\beta = 0$.

In view of 10.10, 10.11 may therefore be written as follows:

10.12 a) $\alpha^0 = 1$,

b) $\alpha^{\beta+} = (\alpha^\beta)\alpha$,

c) $\alpha^\beta = \sup\{\alpha^\gamma : \gamma < \beta\}$,if β is a limit ordinal,

d) $0^\beta = 0$.

We will now develop the fundamental properties of ordinal exponentiation.

10.13 Lemma If γ is a limit ordinal and $\varepsilon < \alpha^\gamma$, then $\exists \pi < \gamma \ni \varepsilon < \alpha^\pi$.

Proof. Suppose, on the contrary, that $\forall \pi < \gamma, \alpha^\pi \leqslant \varepsilon$; this means that ε is an upper bound of the set $\{\alpha^\pi : \pi < \gamma\}$; but α^γ is the sup of this same set, so $\alpha^\gamma \leqslant \varepsilon$, which contradicts our assumption that $\varepsilon < \alpha^\gamma$. Thus for some $\pi < \gamma$, $\varepsilon < \alpha^\pi$. ■

10.14 Theorem For any ordinal numbers $\alpha > 1, \beta, \gamma$,

i) $\beta < \gamma \Rightarrow \alpha^\beta < \alpha^\gamma$,

ii) $\alpha^\gamma < \alpha^\beta \Rightarrow \gamma < \beta$.

Proof

i) The proof is by induction of γ. If $\gamma = 0$, the condition is satisfied vacuously. Now suppose that (i) holds $\forall \delta < \gamma$, that is,

$$\beta < \delta \Rightarrow \alpha^\beta < \alpha^\delta \quad \text{for every} \quad \delta < \gamma.$$

Suppose first that γ is a limit ordinal. If $\beta < \gamma$, then $\beta + 1 < \gamma$, and by the hypothesis of induction we have

$$\beta < \beta + 1 \Rightarrow \alpha^{\beta} < \alpha^{\beta+1};$$

but $\alpha^{\gamma} = \sup\{\alpha^{\beta} : \beta < \gamma\}$, so $\alpha^{\beta+1} \leqslant \alpha^{\gamma}$; thus $\alpha^{\beta} < \alpha^{\gamma}$.

Now suppose that γ is a nonlimit ordinal, $\gamma = \delta + 1$. If $\beta < \gamma$, then $\beta = \delta$ or $\beta < \delta$. If $\beta = \delta$, then we have

$$\begin{aligned} \alpha^{\gamma} = \alpha^{\delta+1} &= \alpha^{\delta}\alpha && \text{by 10.12(b)} \\ &> \alpha^{\delta} && \text{by 9.16(v) (note that } \alpha > 1) \\ &= \alpha^{\beta}. \end{aligned}$$

If $\beta < \delta$, then by the hypothesis of induction, $\alpha^{\beta} < \alpha^{\delta}$, so we have

$$\begin{aligned} \alpha^{\beta} < \alpha^{\delta} &< \alpha^{\delta}\alpha && \text{by 9.16(v)} \\ &= \alpha^{\delta+1} = \alpha^{\gamma} && \text{by 10.12(b).} \end{aligned}$$

ii) $\alpha^{\gamma} \leqslant \alpha^{\beta} \Rightarrow \gamma \leqslant \beta$ is the contrapositive of (i). Now suppose $\alpha^{\gamma} < \alpha^{\beta}$; if $\gamma = \beta$, then $\alpha^{\gamma} = \alpha^{\beta}$, hence $\gamma < \beta$. ■

10.15 Theorem For any ordinal numbers α, β, and γ,

i) $\alpha \leqslant \beta \Rightarrow \alpha^{\gamma} \leqslant \beta^{\gamma}$,

ii) $\beta^{\gamma} < \alpha^{\gamma} \Rightarrow \beta < \alpha$.

Proof

i) The proof is by induction on γ. If $\gamma = 0$, the condition is satisfied trivially. Now suppose that (i) holds $\forall \delta < \gamma$, that is,

$$\alpha \leqslant \beta \Rightarrow \alpha^{\delta} \leqslant \beta^{\delta} \quad \text{for every } \delta < \gamma.$$

Suppose first that γ is a nonlimit ordinal, $\gamma = \delta + 1$. We assume $\alpha \leqslant \beta$ and, by the hypothesis of induction, $\alpha^{\delta} \leqslant \beta^{\delta}$. Thus, by 10.12(b) and 9.16(v) and (vii), we have

$$\alpha^{\gamma} = \alpha^{\delta+1} = \alpha^{\delta}\alpha \leqslant \alpha^{\delta}\beta \leqslant \beta^{\delta}\beta = \beta^{\delta+1} = \beta^{\gamma}.$$

Next, suppose that γ is a limit ordinal; then $\alpha^{\gamma} = \sup\{\alpha^{\delta} : \delta < \gamma\}$. If $\delta < \gamma$, then by the hypothesis of induction $\alpha^{\delta} \leqslant \beta^{\delta}$; but $\beta^{\delta} \leqslant \beta^{\gamma}$ because $\beta^{\gamma} = \sup\{\beta^{\delta} : \delta < \gamma\}$, hence $\alpha^{\delta} \leqslant \beta^{\gamma}$ for every $\delta < \gamma$. It follows that β^{γ} is an upper bound of $\{\alpha^{\delta}: \delta < \gamma\}$, hence $\alpha^{\gamma} \leqslant \beta^{\gamma}$.

ii) This is simply the contrapositive of (i). ■

10.16 Theorem $\alpha^{\beta}\alpha^{\gamma} = \alpha^{\beta+\gamma}$ for any ordinal numbers α, β, and γ.

Proof. The proof is by induction on γ ; the theorem holds trivially if $\gamma = 0$, hence we assume that $\alpha^{\beta}\alpha^{\delta} = \alpha^{\beta+\delta}$ for every ordinal $\delta < \gamma$.

i) Let us suppose first that γ is a nonlimit ordinal, $\gamma = \delta + 1$; then

$$\alpha^{\beta+\gamma} = \alpha^{\beta+\delta+1} = \alpha^{\beta+\delta}\alpha = \alpha^{\beta}\alpha^{\delta}\alpha = \alpha^{\beta}\alpha^{\delta+1} = \alpha^{\beta}\alpha^{\gamma}.$$

ii) Now let us suppose that γ is a limit ordinal; we shall prove the two inequalities (a) $\alpha^{\beta+\gamma} \leqslant \alpha^{\beta}\alpha^{\gamma}$ and (b) $\alpha^{\beta}\alpha^{\gamma} \leqslant \alpha^{\beta+\gamma}$.

a) If γ is a limit ordinal, then clearly $\beta + \gamma$ is a limit ordinal, hence by 10.12(c),

$$\alpha^{\beta+\gamma} = \sup\{\alpha^{\delta} : \delta < \beta + \gamma\}.$$

Now if $\delta < \beta + \gamma$, then either $\delta \leqslant \beta$,or if $\delta > \beta$, then by 9.15, $\delta = \beta + \rho$ for some $\rho < \gamma$ [$\rho < \gamma$ because $\delta = \beta + \rho < \beta + \gamma \Rightarrow \rho < \gamma$ by 9.16(ii)]. In the first case, namely $\delta \leqslant \beta$, it follows by 10.14(i) and 9.16(v) that $\alpha^{\delta} \leqslant \alpha^{\beta} \leqslant \alpha^{\beta}\alpha^{\gamma}$. (We assume that $\alpha \neq 0$, hence $1 \leqslant \alpha^{\gamma}$;if $\alpha = 0$, then 10.16 holds trivially.) In the second case, namely $\delta = \beta + \rho$ where $\rho < \gamma$, it follows by 10.14(i) that $\alpha^{\rho} < \alpha^{\gamma}$, hence by the hypothesis of induction and 9.16(v),

$$\alpha^{\delta} = \alpha^{\beta+\rho} = \alpha^{\beta}\alpha^{\rho} < \alpha^{\beta}\alpha^{\gamma}.$$

Thus, in either of the two cases, $\alpha^{\delta} \leqslant \alpha^{\beta}\alpha^{\gamma}$ for every $\delta < \beta + \gamma$, so $\alpha^{\beta}\alpha^{\gamma}$ is an upper bound of $\{\alpha^{\delta} : \delta < \beta + \gamma\}$, hence $\alpha^{\beta+\gamma} \leqslant \alpha^{\beta}\alpha^{\gamma}$.

b) We are assuming that γ is a limit ordinal; it follows very easily (see Exercise 1(a) at the end of this section) that α^{γ} is a limit ordinal, hence by 10.9(c),

$$\alpha^{\beta}\alpha^{\gamma} = \sup\{\alpha^{\beta}\varepsilon : \varepsilon < \alpha^{\gamma}\}.$$

Now if $\varepsilon < \alpha^{\gamma}$, then by 10.13, $\exists\pi < \gamma\ \varepsilon < \alpha^{\pi}$; hence by 9.16(v), 10.14(i), and the hypothesis of induction,

$$\alpha^{\beta}\varepsilon < \alpha^{\beta}\alpha^{\pi} = \alpha^{\beta+\pi} < \alpha^{\beta+\gamma}.$$

Thus $\alpha^{\beta+\gamma}$ is an upper bound of $\{\alpha^{\beta}\varepsilon : \varepsilon < \alpha^{\gamma}\}$, hence $\alpha^{\beta}\alpha^{\gamma} \leqslant \alpha^{\beta+\gamma}$. ■

10.17 Theorem $(\alpha^{\beta})^{\gamma} = \alpha^{\beta\gamma}$ for any ordinals α, β, and γ.

Proof. The proof is by induction on γ. If $\gamma = 0$, the theorem follows trivially from 10.12(a). Let us assume, then, that $(\alpha^{\beta})^{\delta} = \alpha^{\beta\delta}$ for every $\delta < \gamma$.

i) Suppose first that γ is a nonlimit ordinal, $\gamma = \delta + 1$; then by 10.12(b), the hypothesis of induction, and 10.16, we have

$$(\alpha^{\beta})^{\gamma} = (\alpha^{\beta})^{\delta+1} = (\alpha^{\beta})^{\delta}\alpha^{\beta} = \alpha^{\beta\delta}\alpha^{\beta} = \alpha^{\beta\delta+\beta} = \alpha^{\beta(\delta+1)} = \alpha^{\beta\gamma}.$$

ii) Next, we shall suppose that γ is a limit ordinal and we will prove the two inequalities (a) $(\alpha^{\beta})^{\gamma} \leqslant \alpha^{\beta\gamma}$ and (b) $\alpha^{\beta\gamma} \leqslant (\alpha^{\beta})^{\gamma}$.

a) $$\begin{aligned}(\alpha^{\beta})^{\gamma} &= \sup\{(\alpha^{\beta})^{\delta} : \delta < \gamma\} && \text{by 10.12(c)}\\ &= \sup\{\alpha^{\beta\delta} : \delta < \gamma\} && \text{by the hypothesis of induction.}\end{aligned}$$

But if $\delta < \gamma$, then $\beta\delta < \beta\gamma$, so by 10.14(i), $\alpha^{\beta\delta} < \alpha^{\beta\gamma}$; it follows that $\alpha^{\beta\gamma}$ is an upper bound of $\{\alpha^{\beta\delta} : \delta < \gamma\}$, so $(\alpha^{\beta})^{\gamma} \leqslant \alpha^{\beta\gamma}$.

b) If γ is a limit ordinal, then (see Exercise 6, Exercise Set 9.3) $\beta\gamma$ is a limit ordinal; thus by 10.12(c),

$$\alpha^{\beta\gamma} = \sup\{\alpha^{\delta} : \delta < \beta\gamma\}.$$

Now if $\delta < \beta\gamma$, then by 9.18, $\delta = \beta\xi + \varepsilon$, where $\xi < \gamma$ and $\varepsilon < \beta$; thus

$$\delta = \beta\xi + \varepsilon < \beta\xi + \beta = \beta(\xi + 1).$$

Now $\xi < \gamma$ and γ is a limit ordinal, so $\xi + 1 < \gamma$; thus, by the hypothesis of induction of 10.14(i),

$$\alpha^{\beta(\xi+1)} = (\alpha^{\beta})^{\xi+1} < (\alpha^{\beta})^{\gamma}.$$

Thus, using 10.14(i) once again, $\alpha^{\delta} < \alpha\ \beta(\xi^{+1}) < (\alpha^{\beta})\ ^{\gamma}$. It follows that $(\alpha^{\beta})^{\gamma}$ is an upper bound of $\{\alpha^{\delta} : \delta < \beta\gamma\}$, so $\alpha^{\beta\gamma} \leqslant (\alpha^{\beta})^{\gamma}$. ■

Note that by 10.12(c), $2^{\omega} = \sup\{2^{n} : n < \omega\} = \omega$. We noted in the preceding chapter that the "usual" arithmetic laws do not *all* apply to transfinite ordinal numbers; in particular, the commutative law for multiplication does not hold, nor does the right distributive law. As we shall now see, the law $(\alpha\beta)^{\gamma} = \alpha^{\gamma}\beta^{\gamma}$ does not apply generally to ordinal numbers.

10.18 Example

i) $(2 \cdot 2)^{\omega} = 4^{\omega} = \omega$ because $4^{\omega} = \sup\{4^{n} : n < \omega\} = \omega$.

ii) $2^{\omega}2^{\omega} = \omega\omega = \omega^{2}$. Since $\omega < 1$, it follows by 9.16(v) that $\omega\omega > \omega 1 = \omega$; thus $(2 \cdot 2)^{\omega} \neq 2^{\omega}2^{\omega}$.

EXERCISES 10.2

1. Prove the following:
   a) If γ is a limit ordinal and $\alpha > 1$, then α^{γ} is a limit ordinal.
   b) If α is a limit ordinal and $\gamma \neq 0$, then α^{γ} is a limit ordinal.
2. Prove that for any ordinals $\alpha > 1, \beta$, and γ, $\alpha^{\beta} = \alpha^{\gamma} \Rightarrow \beta = \gamma$.
3. Use 10.5, 10.9, and 10.12 to prove that for any finite ordinal n.
   a) $n + \omega = \omega$, b) $n\omega = \omega$, c) $n^{\omega} = \omega$.
4. Use induction to prove that for every ordinal number β, $2^{\beta} \geqslant \beta$. Consider that for every $\alpha > 0$ and β, $\alpha^{\beta} \geqslant \beta$.
5. Prove that if $\alpha > 1$ and $\beta \neq 0$, then $\alpha\beta \leqslant \alpha^{\beta}$.

6. Prove that if α is a limit ordinal and p and q are finite ordinals, then $(\alpha p)^q = \alpha^q p$. [*Hint*: Use Exercise 8, Exercise Set 9.3.]
7. Let a limit ordinal γ be called *simple* if it cannot be written $\gamma = \delta + \omega$ for any ordinal δ. Prove that γ is simple if and only if $\forall\varepsilon < \gamma$, there exists a limit ordinal λ $\varepsilon < \lambda < \gamma$.
8. If α is a denumerable ordinal (that is, if $\#\alpha = \aleph_0$), use 9.8 to prove that $\alpha\omega$ is denumerable. Conclude that the ordinals $\omega, \omega^2, \ldots, \omega^n, \ldots$ (n finite) are all denumerable.
9. a) Let $\{\gamma_i : i \in I\}$ be a set of ordinals; prove that

$$\sup\{\gamma_i : i \in I\} = \bigcup_{i \in I} \gamma_i.$$

[Use 9.25 and 9.26(ii).]

b) Prove that if $\{\gamma_n : n \in \alpha\}$ is a set of denumerable ordinals, where α is a denumerable, then $\sup\{\gamma_n : n \in \alpha\}$ is a denumerable ordinal. *Note*: It follows immediately from Exercises 2 and 9, Exercise Set 8.5 that

$$\#\bigcup_{i \in I} \gamma_i \leqslant \sum_{i \in I}(\#\gamma_i) \leqslant \aleph_0.$$

c) Use the result of Exercise 8 above to prove that ω^ω is a denumerable ordinal. (Note that $\omega^\omega = \sup\{\omega^n : n \in \omega\}$.)

d) Conclude similarly that ω^{ω^ω}, $\omega^{\omega^{\omega^\omega}}$, etc., are denumerable ordinals.

10. Prove that if α and β are denumerable ordinals, then α^β is a denumerable ordinal. [Use 9(ii) above.]

3 NORMAL FORM

It is a well-known fact of elementary number theory that every natural number n has a uniquely determined decimal representation. That is, given n, there exist unique natural numbers k, $m_0, m_1, \ldots, m_k$ (each $m_i < 10$) such that

$$n = m_k \cdot 10^k + \cdots + m_1 \cdot 10 + m_0.$$

The decimal representation of n is also called its representation with base 10; it can easily be shown that every natural number n also has a unique representation with base b, for any $b > 1$.

The idea of giving every number a representation with base b can easily be extended to ordinal numbers generally. In the sequel we will only consider base ω, but it should be clear to the reader that any other base will do.

10.19 Definition Let γ be an ordinal number; suppose there are nonzero natural numbers $a_1, \ldots, a_n$ and ordinal $\alpha_1 > \alpha_2 > \ldots > \alpha_n$ such that

10.20
$$\gamma = \omega^{\alpha_1} a_1 + \omega^{\alpha_2} a_2 + \cdots + \omega^{\alpha_n} a_n.$$

Then 10.20 is called a *normal form representation* of γ.

10.21 Theorem Every ordinal $\gamma \neq 0$ has a normal form representation.

Proof. The proof will be by induction on γ. If $\gamma = 1$, then $\gamma = \omega^0 1$ is a normal form representation of γ. Now assume the theorem is true $\forall \rho < \gamma$. Let $A = \{\mu : \omega^\mu > \gamma\}$; A is nonempty, as may easily be seen by using Exercise 4, Exercise Set 10.2. Thus A has a least element ν; $\omega^\nu > \gamma$. Suppose ν is a limit ordinal, $\omega^\nu = \sup\{\omega^\delta : \delta < \nu\}$. For each $\delta < \nu, \omega^\delta \leqslant \gamma$ because ν is the *least* element of A; thus γ is an upper bound of $\{\omega^\delta : \delta < \nu\}$,so $\omega^\nu \leqslant \gamma$. This is contrary to our choice of ν, hence $\nu = \alpha_1 + 1$ for some ordinal α_1.

By 9.19, $\gamma = \omega^{\alpha_1}\xi + \rho$, where $\rho < \omega^{\alpha_1}$; clearly $\xi < \omega$, for if $\xi \leqslant \omega$ then

$$\omega^{\alpha_1}\xi \geqslant \omega^{\alpha_1}\omega = \omega^\nu > \gamma,$$

which is impossible by 9.14(i). It follows that ξ is a natural number a_1,so $\gamma = \omega^{\alpha_1}a_1 + \rho$, where $\rho < \omega^{\alpha_1} \leqslant \gamma$. By the hypothesis of induction, there exist nonzero natural numbers $a_2, \ldots, a_n$ and ordinals $\alpha_2 > \ldots > \alpha_n$ such that

$$\rho = \omega^{\alpha_2}a_2 + \cdots + \omega^{\alpha_n}a_n;$$

clearly $\alpha_1 > \alpha_2$, for $\alpha_1 \leqslant \alpha_2$, then

$$\omega^{\alpha_1} \leqslant \omega^{\alpha_2} \leqslant \omega^{\alpha_2}a_2 \leqslant \rho,$$

which is false because $\rho < \omega^{\alpha_1}$. Thus,

$$\gamma = \omega^{\alpha_1}a_1 + \omega^{\alpha_2}a_2 + \cdots + \omega^{\alpha_n}a_n$$

where $a_1, \ldots, a_n$ are nonzero natural numbers and $\alpha_1 > \ldots > \alpha_n$. ■

We will prove next that the normal form representation of γ is unique.

10.22 Lemma If $\gamma = \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_n}a_n$ is a normal form representation of $\gamma < \omega^{\alpha_1+1}$,

Proof. By 10.19, $\alpha_i < \alpha_1$ for $i = 2, \ldots, n$; thus $\omega^{\alpha_i} < \omega^{\alpha_1}$ for $i = 2, \ldots, n$. Thus,

$$\begin{aligned}\gamma = \omega^{\alpha_1}a_1 + \omega^{\alpha_2}a_2 + \cdots + \omega^{\alpha_n}a_n &\leqslant \omega^{\alpha_1}a_1 + \omega^{\alpha_1}a_2 + \cdots + \omega^{\alpha_1}a_n \\ &= \omega^{\alpha_1}(a_1 + \cdots + a_n) < \omega^{\alpha_1}\omega = \omega^{\alpha_1+1}. \ ■\end{aligned}$$

10.23 Theorem The normal form representation of any ordinal γ is unique.

Proof. Suppose $\gamma = \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_n}a_n$, where $a_1, \ldots, a_n, b_1, \ldots, b_m$ are nonzero natural numbers, $\alpha_1 > \alpha_2 > \ldots > \alpha_n$ and $\beta_1 > \ldots > \beta_m$. Let us write

$$\rho_a = \omega^{\alpha_2}a_2 + \cdots + \omega^{\alpha_n}a_n \quad \text{and} \quad \rho_b = \omega^{\beta_2}b_2 + \cdots + \omega^{\beta_m}b_m;$$

thus,

$$\gamma = \omega^{\alpha_1}a_1 + \rho_a = \omega^{\beta_1}b_1 + \rho_b.$$

Suppose $\alpha_1 < \beta_1$; then $\alpha_1 + 1 \leqslant \beta_1$, so

$$\omega^{\alpha_1+1} \leqslant \omega^{\beta_1} \leqslant \omega^{\beta_1}b_1 \leqslant \gamma.$$

But by 10.22, $\gamma < \omega^{\alpha_1+1}$, so we have a contradiction. Analogously, we cannot have $\beta_1 < \alpha_1$, hence $\alpha_1 = \beta_1$. Thus,

$$\gamma = \omega^{\alpha_1}a_1 + \rho_a = \omega^{\alpha_1}b_1 + \rho_b.$$

Now suppose $a_1 < b_1$, hence $a_1 + 1 \leqslant b_1$; then

$$\omega^{\alpha_1}(a_1 + 1) \leqslant \omega^{\alpha_1}b_1,$$

that is,

$$\omega^{\alpha_1}a_1 + \omega^{\alpha_1} \leqslant \omega^{\alpha_1}b_1.$$

Thus

$$\omega^{\alpha_1}a_1 + \omega^{\alpha_1} + \rho_b \leqslant \omega^{\alpha_1}b_1 + \rho_b = \gamma.$$

This gives us $\omega^{\alpha_1}a_1 + \omega^{\alpha_1} + \rho_b \leqslant \omega^{\alpha_1}a_1 + \rho_a$, hence $\omega^{\alpha_1} + \rho_b \leqslant \rho_a$, so $\omega^{\alpha_1} \leqslant \rho_a$. But this is impossible, for $\alpha_2 < \alpha_1$, hence $\omega^{\alpha_2+1} \leqslant \omega^{\alpha_1}$, so by 10.22, $\rho_a < \omega^{\alpha_2+1} \leqslant \omega^{\alpha_1}$. Consequently, we cannot have $a_1 < b_1$; analogously, we cannot have $b_1 < a_1$, so $a_1 = b_1$.

Thus, $\gamma = \omega^{\alpha_1}a_1 + \rho_a = \omega^{\alpha_1}a_1 + \rho_b$, so $\rho_a = \rho_b$. By induction, we may now assume that the normal form representation of $\rho_a = \rho_b$ is unique, hence $\alpha_2 = \beta_2, \ldots, \alpha_n = \beta_n, a_2 = b_2, \ldots, a_n = b_n$. ■

The theorem which follows makes it easy to add and multiply ordinal numbers when they are written in normal form.

10.24 Theorem

i) If $\alpha < \beta$, then $\omega^\alpha a + \omega^\beta b = \omega^\beta b$.

ii) If $\gamma = \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_n}a_n$ is the normal form representation of γ and if $\beta \neq 0$, then $\gamma\omega^\beta = \omega^{\alpha_1+\beta}$

iii) If $\gamma = \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_n}a_n$ is the normal form of γ and if b is finite, then

$$\gamma b = \omega^{\alpha_1} a_1 b + \omega^{\alpha_2} a_2 + \cdots + \omega^{\alpha_n} a_n.$$

Proof

i) If $\alpha < \beta$, then by 9.15, $\beta = \alpha + \delta$ for some $\delta > 0$. Thus,

$$\omega^\alpha a + \omega^\beta b = \omega^\alpha a + \omega^{\alpha+\delta} b = \omega^\alpha a + \omega^\alpha \omega^\delta b = \omega^\alpha (a + \omega^\delta b).$$

Since a is finite, it is clear that $a + \omega^\delta b = \omega^\delta b$ (see, for example, Exercise 3, Exercise Set 10.2). Thus

$$\omega^\alpha a + \omega^\beta b = \omega^\alpha (a + \omega^\delta b) = \omega^\alpha (\omega^\delta b) = \omega^{\alpha+\delta} b = \omega^\beta b.$$

ii) It can be proven very easily that $n\omega = \omega$ for every $n \in \omega$ (see, for example, Exercise 3, Exercise Set 10.2). It follows that $n\omega^\beta = \omega^\beta$ for every $\beta > 0$ and $n \in \omega$. Indeed, for $\beta = 1$ this has just been given: if $\beta > 1$, then by 9.14(i), $\beta = 1 + \delta$ for some $\delta > 0$, so we have

$$n\omega^\beta = n\omega^{1+\delta} = n(\omega\omega^\delta) = (n\omega)\omega^\delta = \omega\omega^\delta = \omega^{1+\delta} = \omega^\beta.$$

Thus

$$\begin{aligned}
\gamma\omega^\beta &= (\omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n)\omega^\beta \\
&\leqslant (\omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_1} a_n)\omega^\beta \\
&= \omega^{\alpha_1}(a_1 + \cdots + a_n)\omega^\beta \\
&= \omega^{\alpha_1} m\omega^\beta && \text{where } m = a_1 + \cdots + a_n \text{ is finite} \\
&= \omega^{\alpha_1}\omega^\beta && \text{because } m\omega^\beta = \omega^\beta \text{, as above} \\
&= \omega^{\alpha_1+\beta}.
\end{aligned}$$

On the other hand, $\omega^{\alpha_1} \leqslant \omega^{\alpha_1} a_1 \leqslant \gamma$, hence $\omega^{\alpha_1+\beta} = \omega^{\alpha_1}\omega^\beta \leqslant \gamma\omega^\beta$.

iii) The proof is by finite induction on b. If $b = 1$, there is nothing to prove. Now suppose (iii) holds for b, and let us prove it for $b + 1$. We have

$$\begin{aligned}
(\omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n)(b+1) &= (\omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n)b \\
&\quad + (\omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n) \\
&= (\omega^{\alpha_1} a_1 b + \omega^{\alpha_2} a_2 + \cdots + \omega^{\alpha_n} a_n) \\
&\quad + (\omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n)
\end{aligned}$$

by the hypothesis of induction. The reader should note that in the above sum, the terms $\omega^{\alpha_2} a_2, \ldots, \omega^{\alpha_n} a_n$ all precede the term $\omega^{\alpha_1} a_1$, and that $\alpha_1 > \alpha_2$, …, $\alpha_1 > \alpha_n$; thus, by 10.24(i), they disappear from the sum. Thus,

$$\begin{aligned}
(\omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n)(b+1) &= \omega^{\alpha_1} a_1 b + \omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n \\
&= \omega^{\alpha_1} a_1 (b+1) + \omega^{\alpha_2} a_2 + \cdots + \omega^{\alpha_n} a_n. \quad \blacksquare
\end{aligned}$$

When adding or multiplying ordinal numbers in normal form, it is also useful to remember that if β is an infinite ordinal and n is finite, then $n + \beta = \beta$ (see, for example, Exercise 3, Exercise Set 10.2).

10.25 Example Let

$$\alpha = \omega^{\omega 2}8 + \omega^{\omega+1}2 + \omega^3 4 \quad \text{and} \quad \beta = \omega^{\omega+3}7 + \omega^{10}5 + 2.$$

We shall form the sums $\alpha + \beta$ and $\beta + \alpha$ and the products $\alpha\beta$ and $\beta\alpha$.

$$\begin{aligned}\alpha + \beta &= \omega^{\omega 2}8 + \omega^{\omega+1}2 + \omega^3 4 + \omega^{\omega+3}7 + \omega^{10}5 + 2\\ &= \omega^{\omega 2}8 + \omega^{\omega+3}7 + \omega^{10}5 + 2.\end{aligned}$$

Note that the terms $\omega^{\omega+1}2$ precede $\omega^{\omega+3}7$,, so by 10.24(i) they disappear from the sum.

$$\begin{aligned}\beta + \alpha &= \omega^{\omega+3}7 + \omega^{10}5 + 2 + \omega^{\omega 2}8 + \omega^{\omega+1}2 + \omega^3 4\\ &= \omega^{\omega 2}8 + \omega^{\omega+1}2 + \omega^3 4 = \alpha.\end{aligned}$$

$$\alpha\beta = \alpha\omega^{\omega+3}7 + \alpha\omega^{10}5 + \alpha 2 \qquad \text{by 9.9(iii)}$$

$$= \omega^{\omega 2+\omega+3}7 + \omega^{\omega 2+10}5 + (\omega^{\omega 2}16 + \omega^{\omega+1}2 + \omega^3 4) \quad \text{by 10.24(ii)–(iii)}$$

$$= \omega^{\omega 3+3}7 + \omega^{\omega 2+10}5 + \omega^{\omega 2}16 + \omega^{\omega+1}2 + \omega^3 4$$

$$\beta\alpha = \beta\omega^{\omega 2}8 + \beta\omega^{\omega+1}2 + \beta\omega^3 4 \qquad \text{by 9.9(iii)}$$

$$= \omega^{\omega+3+\omega 2}8 + \omega^{\omega+3+\omega+1}2 + \omega^{\omega+3+3}4 \qquad \text{by 10.24(ii)}$$

EXERCISES 10.3

1. In each of the following, compute $\alpha + \beta$, $\beta + \alpha$, $\alpha\beta$ and $\beta\alpha$.
   a) $\alpha = \omega^{\omega 3}2 + \omega^{\omega}4 + \omega^{10}5$; $\beta = \omega^{\omega+1}7 + \omega^2 9 + 14$.
   b) $\alpha = \omega^{\omega\omega}9 + \omega^{\omega}7 + \omega 2$; $\beta = \omega^{\omega 5}8 + \omega^7 2 + 1$.
   c) $\alpha = \omega^{\omega 3}\,22 + \omega^{\omega 18}4 + 71$; $\beta = \omega^{\omega 2+3}12 + 100$.
2. Let γ be an ordinal number; prove that γ is irreducible (See Exercise 2, Exercise Set 9.3.) if and only if $\gamma = \omega^{\beta}$ for some ordinal β.

In each of the following exercise, we will assume that

$$\gamma = \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_n}a_n$$

is the normal form representation of γ

3. Prove that γ is a limit ordinal if and only if $\alpha_n \neq 0$.
4. Let us define the *magnitude* of γ to be the ordinal α_1. Prove that $\alpha + \beta = \beta$ if and only if magnitude $\alpha <$ magnitude β.
5. Prove that $\omega\gamma = \gamma$ if and only if $\alpha_1, \ldots, \alpha_n$ are all infinite ordinals. Conclude that $\omega\gamma = \gamma$ if and only

if $\gamma = \omega^{\omega}\beta$ for some ordinal β.

6. Let γ be a limit ordinal and let b be a finite ordinal. Use finite induction on b to prove that

$$\gamma^{b} = \omega^{\alpha_1 b}a_1 + \omega^{\alpha_1(b-1)}[\omega^{\alpha_2}a_2 + \cdots + \omega^{\alpha_n}a_n].$$

7. Use the result of Exercise 6 above to compute α^6 and β^{15}, where α and β are given in Exercise 1(a).

4

Cantor investigated the properties of an interesting class of limit ordinals which he called *epsilon numbers*. These numbers shed some light on the structure of the well-ordered class OR and have useful applications in analysis and elsewhere. We shall give a brief review of their properties in this section.

10.26 Definition Let α be an ordinal number; α is called an *epsilon number* if $\alpha = \omega^{\alpha}$.

It is immediate that every epsilon number is necessarily a limit ordinal. Now, the first question we are led to ask is: Are there any epsilon numbers? To answer this question, we first need a lemma.

10.27 Lemma Let $\{\beta_i : i \in I\}$ be a set of ordinals, and let $\beta = \sup\{\beta_i : i \in I\}$; then $\alpha^{\beta} = \sup\{\alpha^{\beta_i} : i \in I\};$

Proof.

i) Suppose that $\beta = \beta_i$ for some $i \in I$. Thus, $\forall j \in I, \beta_j \leqslant \beta_i$, hence $\alpha^{\beta_j} \leqslant \alpha^{\beta_i}$. It follows that

$$\alpha^{\beta} = \alpha^{\beta_i} = \sup\{\alpha^{\beta_j} : j \in I\}.$$

ii) Now suppose that $\forall i \in I, \beta \neq \beta_i$; β must be a limit ordinal, for if $\beta = \delta + 1$, then $\exists i \in I\ \beta_i > \delta$,so $\beta_i = \delta + 1 = \beta$, which is contrary to our assumption. Now $\alpha^{\beta} = \sup\{\alpha^{\gamma} : \gamma < \beta\}$; for each $i \in I$, $\beta_i < \beta$, hence $\alpha^{\beta i} < \alpha^{\beta}$; thus $\sup\{\alpha^{\beta_i} : i \in I\} \leqslant \alpha^{\beta}$.. On the other hand, if $\gamma < \beta$, then $\beta_i > \gamma$ for some $i \in I$, hence $\alpha^{\gamma} < \alpha^{\beta_i} \leqslant \sup\{\alpha^{\beta_i} : i \in I\}$.. Thus

$$\alpha^{\beta} = \sup\{\alpha^{\gamma} : \gamma < \beta\} \leqslant \sup\{\alpha^{\beta_i} : i \in I\}.$$

Consequently,

$$\alpha^{\beta} = \sup\{\alpha^{\beta_i} : i \in I\}.$$

We now return to the question: Are there any epsilon numbers? The answer is “yes,” and this can easily be shown as follows:

We define a function $f_0 : \omega \to \mathrm{OR}$ as follows:

$$f_0(0) = 1, \quad f_0(n+1) = \omega^{f_0(n)}.$$

The existence of f_0 is guaranteed by the finite recursion theorem, 6.8. Clearly, we have

$$f_0(0) = 1,\ f_0(1) = \omega,\ f_0(2) = \omega^{\omega},\ f_0(3) = \omega^{\omega^{\omega}}, \qquad \text{and so on.}$$

Now, let $\varepsilon_0 = \sup\{f(n) : n \in \omega\}$; we claim that ε_0 is an epsilon number. Indeed, by 10.27,

$$\omega^{\varepsilon_0} = \sup\{\omega^{f(n)} : n \in \omega\} = \sup\{f(n+1) : n \in \omega\} = \varepsilon_0.$$

Thus there is at least one epsilon number, namely ε_0; we can easily show, in fact, that ε_0 is the least epsilon number.

10.29 ε_0 is the least epsilon number.

Proof. If α is an epsilon number, then $f_0(0) = 1 \leqslant \alpha$ (for clearly 0 is not an epsilon number). Now suppose that $f(n) \leqslant \alpha$; then $f(n + 1) = \omega^{f(n)} \leqslant \omega^{\alpha} = \alpha$. It follows by finite induction that $f(n) \leqslant \alpha$ for every $n \in \omega$; thus, $\varepsilon_0 \leqslant \alpha$. ■

Is ε_0 the only epsilon number or are there others? It is easy to answer this question, for the method we used to construct ε_0 may be used to construct infinitely many other epsilon numbers. Indeed, we have the following:

10.30 If α is any ordinal number, let f_α be the function defined inductively by the pair of conditions

$$f_\alpha(0) = \alpha + 1,$$
$$f_\alpha(n+1) = \omega^{f_\alpha(n)}$$

and let $S(\alpha) = \sup\{f_\alpha(n) : n \in \omega\}$. Then $S(\alpha)$ is an epsilon number; furthermore, $S(\alpha)$ is the least epsilon number greater than α.

Proof. By 10.27,

$$\omega^{S(\alpha)} = \sup\{\omega^{f_\alpha(n)} : n \in \omega\} = \sup\{f_\alpha(n+1) : n \in \omega\} = S(\alpha).$$

Clearly $\alpha > S(\alpha)$. Now let γ be an epsilon number such that $\alpha > \gamma$; then $f_\alpha(0) = \alpha + 1 \leqslant \gamma$. Furthermore, assuming that $f_\alpha(n) \leqslant \gamma$, we have

$$f_\alpha(n+1) = \omega^{f_\alpha(n)} \leqslant \omega^{\gamma} = \gamma.$$

It follows, by finite induction, that $f_\alpha(n) \leqslant \gamma$ for every $n \in \omega$, hence $S(\alpha) \leqslant \gamma$. Thus $S(\alpha)$ is the least epsilon number greater than α.

It is easy to see that ε_0 is $S(0)$; thus, the first few epsilon numbers are $S(0)$, $S(1)$, $S(2)$, $S(3)$, etc. The next question we are led to ask is: Are there any epsilon numbers greater than all the $S(n)$, $(n \in \omega)$? The question is easily answered in the following theorem.

10.31 Let $\{\alpha_i : i \in I\}$ be a set of epsilon numbers; if $\beta = \sup\{\alpha_i : i \in I\}$, then β is an epsilon number.

Proof. By 10.27,

$$\omega^{\beta} = \sup\{\omega^{\alpha_i} : i \in I\} = \sup\{\alpha_i : i \in I\} = \beta.$$

Let $\varepsilon : \mathrm{OR} \to \mathrm{OR}$ be the function defined recursively as follows [we agree to write ε_i for $\varepsilon(i)$]:

10.32

$$\begin{aligned} \varepsilon_0 &= S(0), \\ \varepsilon_{\alpha+1} &= S(\varepsilon_\alpha), \quad \forall \alpha \in \mathrm{OR} \\ \varepsilon_\beta &= \sup\{\varepsilon_\gamma : \gamma < \beta\} \quad \text{if } \beta \text{ is a limit ordinal.} \end{aligned}$$

10.33 ε is an isomorphism from OR to the class of all the epsilon numbers.

Proof. To prove that ε_β is an epsilon number for every $\beta \in \mathrm{OR}$, we argue by induction:

i) We have already seen that ε_0 is an epsilon number.

ii) Now assume that ε_α is an epsilon number for every $\alpha < \beta$. If β is a nonlimit ordinal, $\beta = \delta + 1$, then

$$\varepsilon_\beta = \varepsilon_{\delta+1} = S(\varepsilon_\delta)$$

is an epsilon number by 10.30. If β is a limit ordinal, then

$$\varepsilon_\beta = \sup\{\varepsilon_\gamma : \gamma < \beta\}$$

is an epsilon number by 10.31. Thus, $\forall \beta \in \mathrm{OR}$, ε_β is an epsilon number. It is immediate that ε is a strictly increasing function, hence it is injective. To show that the range of ε is the class E of the epsilon numbers, let $\delta \in E$ and let ε_γ be the least ε_ζ such that $\delta < \varepsilon_\zeta$. Now γ cannot be a limit ordinal, for if it is, then by 10.32,

$$\varepsilon_\gamma = \sup\{\varepsilon_\pi : \pi < \gamma\}$$

hence (because $\delta < \varepsilon_\gamma$) $\exists \pi < \gamma\ \delta < \varepsilon_\pi$, which contradicts our choice of γ. Thus, γ is a nonlimit ordinal, $\gamma = \pi + 1$; by the choice of γ, $\varepsilon_\pi \leqslant \delta > \varepsilon_{\pi+1}$, so by 10.30, $\delta = \varepsilon_\pi$. The fact that ε is an isomorphism follows now by 4.48.

It turns out, then, that there are “as many” epsilon numbers as there are ordinals. The class of the epsilon numbers is

$$\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\omega, \varepsilon_{\omega+1}, \ldots, \varepsilon_{\omega2}, \ldots, \varepsilon_{\omega3}, \ldots, \varepsilon_{\omega\omega}, \ldots, \varepsilon_\alpha, \ldots,$$

as α ranges over all the ordinals.

A few easy arithmetic rules simplify computations which involve epsilon numbers. They are given in the next theorem.

10.34 Let ε designate an arbitrary epsilon number. Then

i) If $\alpha > \varepsilon$, then $\alpha + \varepsilon = \varepsilon$.

ii) If $\alpha > \varepsilon$, then $\alpha\varepsilon = \varepsilon$.

iii) If $\alpha > \varepsilon$, then $\alpha^{\varepsilon} = \varepsilon$.

Proof. Let $\alpha > \varepsilon$, and write α in normal form:

$$\alpha = \omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n.$$

i) $\alpha + \varepsilon = \alpha + \omega^{\varepsilon} = \omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n + \omega^{\varepsilon}$. Now $\alpha < \varepsilon$, hence for $i = 1, \ldots, n$, $\omega^{\alpha_i} < \varepsilon = \omega^{\varepsilon}$, so $\alpha_i < \varepsilon$. It follows by 10.24(i) that

$$\omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n + \omega^{\varepsilon} = \varepsilon, \quad \text{so } \alpha + \varepsilon = \varepsilon.$$

ii)
$$\begin{aligned} \alpha\varepsilon = \alpha\omega^{\varepsilon} &= (\omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n)\omega^{\varepsilon} \\ &= \omega^{\alpha_1 + \varepsilon} && \text{by 10.24(ii)} \\ &= \omega^{\varepsilon} && \text{by 10.34(i)} \\ &= \varepsilon. \end{aligned}$$

iii)
$$\begin{aligned} \alpha &= \omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_n} a_n \leqslant \omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_1} a_n \\ &= \omega^{\alpha_1}(a_1 + \cdots + a_n) < \omega^{\alpha_1}\omega = \omega^{\alpha_1 + 1}. \end{aligned}$$

Thus $\alpha^{\varepsilon} \leqslant (\omega^{\alpha_1+1})^{\varepsilon} = \omega^{(\alpha_1+1)\varepsilon}$.. Since every epsilon number is a limit ordinal, $\alpha_1 < \varepsilon \Rightarrow \alpha_1 + 1 < \varepsilon$, so by 10.34(ii), $(\alpha_1 + 1)\varepsilon = \varepsilon$. Thus, finally,

$$\omega^{(\alpha_1+1)\varepsilon} = \omega^{\varepsilon} = \varepsilon, \quad \text{so } \alpha^{\varepsilon} \leqslant \varepsilon.$$

But we always have $\alpha^{\varepsilon} \geqslant \varepsilon$ (see Exercise 4, Exercise Set 10.2.), so $\alpha^{\varepsilon} = \varepsilon$.

It is worth noting that α is an epsilon number if and only if α satisfies the following condition:

10.35 If $\beta > \alpha$ and $\gamma > \alpha$, then $\beta^{\gamma} > \alpha$.

The proof of this statement is left as an exercise for the reader (Exercise 4 below).

EXERCISES 10.4

1. Prove that if α is an epsilon number, then α is a limit ordinal.
2. If α is an epsilon number, prove that $\beta^{\omega^{\alpha}} = \beta^{\omega\alpha}$ for every ordinal number β.

3. Prove that if α is an infinite ordinal and $\alpha^\beta = \beta$, then β is an epsilon number.
4. a) Prove that if $\alpha(\alpha > \omega)$ satisfies 10.35, then α is a limit ordinal.

   b) Prove that $\alpha(\alpha > \omega)$ is an epsilon number if and only if α satisfies 10.35.
5. Prove that $\alpha(\alpha > \omega)$ is an epsilon number if and only if $2^\alpha = \alpha$.

5 INACCESSIBLE ORDINALS AND CARDINALS

Inaccessible ordinals and cardinals play an important role in current investigations on the axiomatic foundations of set theory. They also have applications in functional analysis, topology, algebra, mathematical logic and other areas of advanced mathematics. In this section we will introduce them and give a few of their basic properties.

If α is any ordinal numbers, let $Z(\alpha)$ designate the class of all the ordinals which are equipotent with α; the least element of $Z(\alpha)$ is called the *initial ordinal belonging to* α, and is denoted by $\mathrm{Io}(\alpha)$. It is immediate that

10.36 $\forall\alpha \in \mathrm{OR}$, $\mathrm{Io}(\alpha) \leqslant \alpha$,

and

10.37 $\mathrm{Io}(\alpha)$ is the largest initial ordinal less than or equal to α.

(To prove 10.37, note that if $\mathrm{Io}(\alpha) > \gamma \geqslant \alpha$, then $\mathrm{Io}(\alpha) \subseteq \gamma \subseteq \alpha$, hence $\mathrm{Io}(\alpha) \approx \alpha \approx \gamma$.) We have seen that the class of all the initial ordinals satisfies conditions K1 and K2 of the axiom of cardinality, hence we are justified in making the following definition (see 9.33):

> By a cardinal number we mean an initial ordinal; thus, the class CD of all the cardinal numbers is the class of all the initial ordinals.

We have noted that if α is any ordinal number, it is common practice to write $\aleph_{0}{}_{\alpha} = \omega_\alpha$, treating $\aleph_{0}{}_{\alpha}$ as a cardinal and ω_α as an ordinal. If α is any ordinal number, it is easy to see that

10.38

$$\mathrm{Io}(\alpha) = \#\alpha.$$

It follows easily from 10.36, 10.37, 10.38 that if A is any well-ordered set,

10.39

$$\oslash A \geqslant \omega_\alpha \quad \text{iff} \quad \#A \geqslant \aleph_\alpha.$$

(To prove 10.39, set $\gamma = \oslash A$ hence $\mathrm{Io}(\gamma) = \#\gamma = \#A$; the simple details are left as an exercise for the reader.)

We will now begin the process of defining inaccessible ordinals and cardinals.

10.40 Definition Let A be a well-ordered class and let $B \subseteq A$; we say that B is a *cofinal subclass* of A if

$$\forall x \in A, \quad \exists y \in B \ni y > x.$$

10.41 Lemma If C is a cofinal subclass of B and B is a cofinal subclass of A, then C is a cofinal subclass of A.

Proof. Let $x \in A$; then $\exists y \in B \in y > x$; hence $\exists z \in C \ni z > y$, so $z > x$.

10.42 Lemma Let α be an ordinal and let $B \subseteq \alpha$; B is a cofinal subset of α if and only if $\sup B = \alpha$.

Proof. The reader should note that by 9.26(ii) and 9.24, every element of an ordinal number is an ordinal number, and $\gamma > \alpha$ iff $\gamma \in \alpha$. ■

i) Let B be a cofinal subset of α;$\forall x \in B$, $x \in \alpha$, that is, $x < \alpha$, so α is an upper bound on B. Now if $\gamma < \alpha$, then $\gamma \in \alpha$, so by 10.40, $\exists \beta \in B \ni \beta > \gamma$, so γ is not an upper bound on B; this proves that α is the *least* upper bound of B.

ii) Suppose $\sup B = \alpha$; if $\gamma \in \alpha$, that is, $\gamma < \alpha$, then (because γ is *not* an upper bound of B) $\exists \beta \in B \ni \beta > \gamma$. Thus B is a cofinal subset of α. ■

10.43 Definition Let α be a limit ordinal; by the *cofinality* of α we mean the least ordinal β such that α has a cofinal subset similar to β. If β is the cofinality of α,we write $\beta = cf\,(\alpha)$.

10.44 If $\beta = cf\,(\alpha)$ for some $\alpha \in$ OR, then $\beta = cf\,(\beta)$.

Proof. Let $\gamma = cf\,(\beta)$; now α has a cofinal subset similar to β and β has a cofinal subset similar to γ, hence by 10.41, α has a cofinal subset similar to γ. But β is the least ordinal γ such that α has a cofinal subset similar to γ, so $\beta \leqslant \gamma$;now γ & $\leqslant \beta$ because γ is similar to a subset of β,so $\beta = \gamma$. ■

10.45 Let β be a limit ordinal; if $\beta = cf\,(\beta)$, then β is an initial ordinal.

Proof. Let $\omega_\alpha = \mathrm{Io}(\beta)$, and let f be a bijective function $f : \omega_\alpha \to \beta$; let $A = \{\gamma \in \omega_\alpha : \forall \delta < \gamma, f(\delta) < f(\gamma)\}$. We will show first that $\bar{f}(A)$ is a cofinal subset of β. Indeed, if $v \in \beta$, then (because β is a limit ordinal and a limit ordinal has no greatest element), there are elements $\xi \in \omega_\alpha$ $f(\xi) > v$. (Remember that f is bijective!) Let π be the least such element; then $\forall \xi < \pi$, $f(\xi) \leqslant v < f(\pi)$, hence $\pi \in A$; this proves that $\forall v \in \beta$, $\exists \pi \in A \ni f(\pi) > v$; thus $\bar{f}(A)$ is a cofinal subset of β.

Because of the way we have defined A, it is easy to see that $f_{[A]} : A \to \bar{f}(A)$ is an isomorphism; thus, if $\delta = \oslash A$, then δ is similar to a cofinal subset of β, so $cf\,(\beta) \leqslant \delta$. But A is a subset of ω_α,so $\delta \leqslant \omega_\alpha$; thus, $\beta = cf\,(\beta) \leqslant \omega_\alpha$. But ω_α is the *least* ordinal equipotent with β, so $\omega_\alpha \leqslant \beta$. Thus $\omega_\alpha = \beta$, so β is an initial ordinal. ■

10.46 A limit ordinal β is called *regular* if $cf\,(\beta) = \beta$.

It follows, by Lemma 10.44, that the class of the regular ordinals is the range of the function cf : OR → OR. By Theorem 10.45, *every regular ordinal is an initial ordinal* ω_α. Thus, in particular, every regular ordinal is a cardinal.

10.47 If ω_α is a regular ordinal, then $\aleph_{0\alpha}$ is called a *regular cardinal*.

The significance of regular cardinals in set theory will become apparent once we have given an alternative definition for them. We begin with the following two lemmas:

10.48 Let $\{\gamma_i : i \in I\}$ be a set of ordinals, and let $\#\gamma_i = \aleph_{0\delta i}$ for each $i \in I$. If $\omega_\alpha = \sup\{\gamma_i : i \in I\}$, then $\alpha = \sup\{\delta_i : i \in I\}$.

Proof. If $\omega_\alpha = \sup\{\gamma_i : i \in I\}$, then, for each $i \in I$, $\gamma_i \leqslant \omega_\alpha$, hence by 10.39, $\aleph_{0\delta i} = \#\gamma_i \leqslant \aleph_{0\alpha}$, so by 9.22, $\delta_i \leqslant \alpha$. Now suppose that for every $i \in I$, $\delta_i < \pi$; then $\forall i \in I$, $\#\gamma_i = \aleph_{0\delta_i} < \aleph_{0\pi}$, hence by 10.39 $\gamma_i > \omega_\pi$. But $\omega_\alpha = \sup\{\gamma_i : i \in I\}$, so $\omega_\alpha \leqslant \omega_\pi$, hence by 9.22, $\alpha \leqslant \pi$. It follows that α is the least upper bound of $\{\delta_i : i \in I\}$.

10.49 Let $\{\delta_i : i \in I\}$ be a set of ordinals, and let α be any ordinal such that $\# I < \aleph_{0\alpha}$. Then $\sum_{i\in I}\aleph_{\delta_i} = \aleph_{0\alpha}$ if and only if $\alpha = \sup\{\delta_i : i \in I\}$.

Proof. If I is finite, the result follows trivially; thus, we may assume I is infinite.

i) Suppose

$$\sum_{i\in I}\aleph_{\delta_i} = \aleph_\alpha.$$

Then for each $j \in I$,

$$\aleph_{\delta_j} \leqslant \sum_{i\in I}\aleph_{\delta_i} = \aleph_\alpha,$$

hence $\delta_j \leqslant \alpha$. Thus α is an upper bound of $\{\delta_i : i \in I\}$. Before going on, we note that there exists an $i \in I$ such that $\#I < \aleph_{0\delta i}$; for if $\aleph_{0\delta i} \leqslant \#I$ for every $i \in I$, then by 8.19, $\sum_{i\in I}\aleph_{\delta_i} \leqslant (\#I)(\#I) = \#I < \aleph_{0\alpha}$, which is contrary to our assumption. Now suppose that $\delta_i < \pi$ for every $i \in I$. Then by 8.19,

$$\sum_{i\in I}\aleph_{\delta_i} \leqslant (\#I)\aleph_\pi.$$

But we have just seen that for some $i \in I$, $\#I < \aleph_{0\delta i}$, hence $\#I \leqslant \aleph_{0\pi}$; thus $(\#I)\aleph_{0\pi} = \aleph_{0\pi}$. It follows that $\sum_{i\in I}\aleph_{\delta_i} \leqslant \aleph_{0\pi}$, that is, $\aleph_{0\alpha} \leqslant \aleph_{0\pi}$, so by 9.22, $\alpha \leqslant \pi$. We have proved that α is the least upper bound of $\{\delta_i : i \in I\}$.

ii) Suppose $\alpha = \sup\{\delta_i : i \in I\}$. For each $i \in I$, $\delta_i \leqslant \alpha$, so by 9.22, $\aleph_{\delta_i} \leqslant \aleph_\alpha$. Thus by 8.19,

$$\sum_{i\in I}\aleph_{\delta_i} \leqslant (\#I)\aleph_\alpha = \aleph_\alpha.$$

Now let $\sum_{i\in I}\aleph_{\delta_i} = \aleph_\delta$. Then for each $j \in I$,

$$\aleph_{\delta_j} \leqslant \sum_{i\in I}\aleph_{\delta_i} = \aleph_\delta,$$

hence by 9.22, $\delta_j \leqslant \delta$. But $\alpha = \sup \delta_i$,so $\alpha \leqslant \delta$, hence

$$\aleph_\alpha \leqslant \aleph_\delta = \sum_{i\in I}\aleph_{\delta_i}.$$

Thus, finally,

$$\sum_{i\in I}\aleph_{\delta_i} = \aleph_\alpha. \ ■$$

10.50 Theorem $\aleph_\alpha$ is a regular cardinal if and only if it satisfies the following condition:

10.51 If $\{a_i : i \in I\}$ is any set of cardinals such that $a_i < \aleph_\alpha$ for each $i \in I$ and $\#I < \aleph_\alpha$, then $\sum_{i\in I} a_i < \aleph_\alpha$.

Proof

i) Suppose $\aleph_\alpha$ is a regular cardinal and $\{a_i : i \in I\}$ is a set of cardinals such that $a_i < \aleph_\alpha$ for each $i \in I$ and $\#I < \aleph_\alpha$. We may assume the a_i are all infinite cardinals, for any finite α_i among them would not affect our result; thus, we may set $a_i = \aleph_{\delta_i}$ for each $i \in I$. We now have $\aleph_{\delta_i} < \aleph_\alpha$ for each $i \in I$ and $\#I < \aleph_\alpha$, so by 8.19, $\sum_{i\in I}\aleph_{\delta_i} \leqslant \aleph_\alpha$. Now if $\sum_{i\in I}\aleph_{\delta_i} = \aleph_\alpha$, then by 10.49, $\alpha = \sup\{\delta_i : i \in I\}$,

hence by 9.22,

$$\omega_\alpha = \sup\{\omega_{\delta_i} : i \in I\},$$

so by 10.42, $\{\omega_{\delta i} : i \in I\}$ is a cofinal subset of ω_α. But this is impossible, for the following reason: $\#I > \aleph_\alpha$, hence

$$\#\{\omega_{\delta_i} : i \in I\} < \aleph_\alpha,$$

so by 10.39, $\circledcirc\{\omega_{\delta_i} : i \in I\} < \omega_\alpha$. which is in contradiction with the fact that ω_α is a regular ordinal.

ii) Conversely, suppose $\aleph_\alpha$ satisfies 10.51, and let $\{\gamma_i : i \in I\}$ be a cofinal subset of ω_α, that is,

$$\omega_\alpha = \sup\{\gamma_i : i \in I\}.$$

[We will assume that the γ_i are all distinct, hence $\#\{\gamma_i : i \in I\} = \#I$.] Set $\aleph_{\delta_i} = \#\gamma_i$ for each $i \in I$. Then, by 10.48, $\alpha = \sup\{\delta_i : i \in I\}$, hence by 10.49,

$$\sum_{i\in I}\aleph_{\delta_i} = \sum(\#\gamma_i) = \aleph_\alpha .$$

Thus, by 10.51, we must necessarily have $\#I \geqslant \aleph_\alpha$, so by 10.39,

$$\oslash\{\gamma_i : i \in I\} \geqslant \omega_\alpha .$$

This proves that any cofinal subset of ω_α has ordinality $\geqslant \omega_\alpha$,so ω_α is a regular ordinal.

10.52 Corollary If α is a nonlimit ordinal, then $\aleph_\alpha$ is a regular cardinal.

Proof. If $\alpha = \delta + 1$, then set $a_i = \aleph_\delta$ for each $i \in I$, where $\#I = \aleph_\delta$. Then by 8.17,

$$\sum_{i\in I} a_i = (\aleph_\delta)^2 = \aleph_\delta < \aleph_{\delta+1} .$$

Thus, by 10.50, $\aleph_\alpha$ is a regular cardinal.

10.53 Definition Let a be a cardinal number; a is called an *inaccessible* cardinal (more precisely, a *strongly inaccessible* cardinal) if

i) a is a regular cardinal, and

ii) $b < a$ and $c < a \Rightarrow b^c < a$.

ω_α is called an *inaccessible ordinal* if $\aleph_\alpha$ is an inaccessible cardinal.

In view of Theorem 10.50, an infinite cardinal number b is inaccessible if and only if it satisfies the following pair of conditions:

i) If $\{a_i : i \in I\}$ is a set of cardinals such that $a_i < b$ for each $i \in I$ and $\#I < b$, then $\sum_{i\in I} a_i < b.$

ii) If $a < b$ and $c < b$, then $a^c < b$.

In other words, an infinite cardinal number b is inaccessible if and only if it cannot be obtained *either as a sum of fewer than b cardinals smaller than b, or by raising a cardinal smaller than b to a power smaller than b*. This explains the use of the word "inaccessible." It can be shown, furthermore, that if b is an inaccessible cardinal, then b cannot be obtained as a product of fewer than b cardinals smaller than b (see Exercise 4, below).

We may call a nonempty set A *inaccessible* if it cannot be constructed from smaller sets by using the set-theoretical operations of union, intersection, product, or power set. To be technical, this means that

i) A is not equal to $\bigcap_{i\in I} B_i$, $\bigcup_{i\in I} B_i$, or $\prod_{i\in I} B_i$ for any set of sets $\{B_i : i \in I\}$, where $\{B_i : i \in I\} \prec A$ and $B_i \prec A$ for each $i \in I$;

ii) A is not the power set of any set B such that $B \prec A$.

For example, if we begin with the empty set and start to construct sets such as

$$\{\varnothing\}, \quad \{\varnothing, \{\{\varnothing\}\}, \{\varnothing\}\}, \quad \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \quad \text{etc.,}$$

and if we proceed to construct larger and larger sets from these by using the operations of union (including infinite union), product (including infinite product) and power set, we will never end up with an inaccessible set. Now, in view of what we have said in the preceding paragraph, it is clear that inaccessible cardinals are the cardinals of inaccessible sets.

A rather obvious question which arises in axiomatic set theory is, "Are there any inaccessible sets?" In other words, do inaccessible cardinals exist? The existence of inaccessible cardinals cannot be proved by means of the axioms we have already introduced; however, a new axiom may be added to set theory, called the *axiom for inaccessible cardinals*, asserting their existence. It has been proven in recent years that this axiom is not a consequence of the other axioms of set theory; whether it is consistent with the other axioms of set theory is still an open question.

The following definition is useful:

10.54 Definition $\aleph_\alpha$ is called a *weakly inaccessible cardinal* if

i) $\aleph_\alpha$ is a regular cardinal,

ii) α is a limit ordinal.

By Corollary 10.52, if α is a nonlimit ordinal, then $\aleph_\alpha$ is necessarily regular; thus, it is natural to ask whether there are any regular cardinals $\aleph_\alpha$ where α is a limit ordinal. Again, we cannot prove the existence of such cardinals from the usual axioms of set theory, but their existence follows immediately from the axiom for inaccessible cardinals; indeed, we have:

10.55 Theorem If *a* is strongly inaccessible, then *a* is weakly inaccessible.

Proof. Suppose $\aleph_\alpha$ is strongly inaccessible, and assume α is a nonlimit ordinal, $\alpha = \delta + 1$. By 8.16,

$$\aleph_\delta^{\aleph_\delta} = 2^{\aleph_\delta} > \aleph_\delta, \quad \text{hence} \quad \aleph_\delta^{\aleph_\delta} \geqslant \aleph_\alpha$$

which is contrary to our hypothesis that $\aleph_\alpha$ is strongly inaccessible; thus α is a limit ordinal. ■

There is another interesting connection between strong and weak inaccessibility:

10.56 Theorem Assuming the generalized continuum hypothesis to be true, if *a* is weakly inaccessible then *a* is strongly inaccessible.

Proof. Assume the generalized continuum hypothesis, let $\aleph_\alpha$ be weakly inaccessible, and suppose $\aleph_\beta < \aleph_\alpha$. By 9.22, $\beta > \alpha$, and since α is a limit ordinal, $\beta + 1 < \alpha$,so $\aleph_{\beta+1} < \aleph_\alpha$. But by the generalized continuum hypothesis, $2^{\aleph_\beta} = \aleph_{\beta+1}$, hence $2^{\aleph_\beta} < \aleph_\alpha$. It follows (see Exercise 3, Exercise Set 10.5.) that $\aleph_\alpha$ is strongly inaccessible. ■

Thus, if we assume the generalized continuum hypothesis, then the notions of strongly inaccessible and weakly inaccessible are equivalent.

EXERCISES 10.5

1. Prove 10.37 and 10.38.
2. Prove 10.39.
3. Prove that a is strongly inaccessible if and only if

   a) a is regular, and

   b) $\forall b < a$, $2^b < a$. [See Exercise 3, Exercise Set 8.4.]
4. Prove that if b is an inaccessible cardinal, than b satisfies the following condition:

   If $\{a_i : i \in I\}$ is a set of cardinals such that $a < b$ for each $i \in I$ and $\#I < b$, then $\prod_{i \in I} a_i < b$

5. Prove that if an infinite cardinal b satisfies the condition in Exercise 4, then b is an inaccessible cardinal.

11

Consistency and Independence in Set Theory

1 WHAT IS A SET?

Most textbooks of set theory begin by asking this question. However, it is only now— at this stage of the course—that it is possible to give a meaningful answer. When Georg Cantor first began writing about sets, he defined a set to be any collection of definite objects. That's the way most people still think of sets, and they're not wrong. In the material world, sets are finite collections of objects, and what is called *naïve set theory* is perfectly adequate and correct for finite sets. It is only in the case of infinite sets that difficulties make their appearance. People have no experience with infinite collections of things, and that is why our intuitions fall short.

Cantor constructed his theory as a theory of *infinite* sets, and fully understood that they are very different from finite sets. He saw no reason to distinguish sets in mathematics from arbitrary collections of objects, because the discovery of the famous "paradoxes", or contradictions, was still decades away. Moreover, the theory he constructed was so beautiful in its clarity and simplicity that when we recollect it today, we refer to it as "Cantor's Paradise". In fact, when the earliest of the paradoxes were announced, nobody took them seriously, because it was so comfortable in Cantor's Paradise that nobody wanted to leave.

As we have seen in previous chapters, in set theory we can construct the integers, and from them we may define rational numbers as ordered pairs of integers. Every real number may be identified with a convergent sequence of rational numbers. There is no need to go on: All of mathematics, piece by piece, can be constructed out of sets. Although sets are viewed differently today, it still remains true that all of mathematics can be built within the framework of set theory.

Axiomatic set theory is not for the masses: It is a specialized branch of mathematics built on firm, solid foundations, designed to be used as a framework for all other mathematical theories. The various axiomatic systems of set theory used today are lean and mean: They assume the bare minimum necessary for the construction of number systems, analysis, and mathematics generally. To understand the motivation behind them, you must keep in mind that they are designed to be minimalist and pragmatic in character: They presuppose the absolute least that suffices to get their enterprise off the ground, and are directed solely toward the goal of constructing mathematical models.

The class OR of all the ordinal numbers is indispensable in mathematics, hence we begin by establishing the existence of OR. The class of the ordinals is defined recursively beginning with the empty set as follows:

$$0 = \emptyset.$$

$$\alpha^+ = \alpha \cup \{\alpha\}.$$

$$\alpha = \bigcup_{\beta<\alpha} \beta \quad \text{if } \alpha \text{ is not an immediate successor.}$$

As noted in Chapter 9, the class OR of the ordinals is a proper class.

Inspired by the simplicity of the class OR, Zermelo defined a hierarchy of sets which—he proposed—was adequate for carrying out all of mathematics. It is known as the *Cumulative Hierarchy*, because its members are arranged in successive levels indexed by the ordinal numbers.

$$V_0 = \emptyset$$
$$V_{\alpha+1} = \mathscr{P}(V_\alpha)$$
$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Finally, V is defined to be the union of all the levels: $V = \bigcup_{\alpha \in OR} V_\alpha$

This construction begins with just one set, which is the empty set. Then, by using the power set axiom and the axiom of unions, we build progressively larger sets. Note that these are the only sets that are *guaranteed* to exist in axiomatic set theory. The essence of the construction process is that we begin at a stage 0 with the simplest possible set, which is the empty set, and work upward stage by stage. At every stage, let us say stage $\alpha + 1$, the elements of $V_{\alpha+1}$ are all the sets in V_α (that is, the subsets of V_α). And if α is a limit ordinal, then all the elements of V_α are sets that exist at the previous levels V_β for $\beta < \alpha$. So the *elements* at each level are the *sets* that were created at the previous levels.

The value of this hierarchy is that it classifies sets according to their complexity, that is, their "distance" up from the empty set. The division of all the sets into successive levels means that you can prove things by transfinite induction, and define objects using transfinite recursion. The "distance of a set from the empty set" is called its *rank* and defined as follows:

11.1 Definition The *rank* of a set A is the least ordinal α such that $A \in V_\alpha$.

Zermelo proposed that for the purposes of mathematics every possible set is a member of V, and he called V the *universe of sets*. In order for this claim to be plausible, it must be possible to show that the members of V satisfy all the axioms of set theory. To do this, we must first show that the members of V have two essential properties which they share (almost) with the ordinals. The first property is given in the two definitions that follow:

11.2 Definition If we treat the membership relation $\in$ as a relation on a set A, then an element $x \in A$ is said to be $\in$-*minimal* if there is no element $y \in A$ such that $y \in x$.

It is very important, here, to distinguish a minimal element of A from a least element of A: The least element is comparable with every element of A. But a minimal element is not.

11.3 DefinitionA set A is said to be *well-founded* if every non-empty subset of A has an $\in$-minimal element.

11.4 *Remark.*Note that a least element of a set is always minimal, but the converse is not true. Thus, if a set A is $\in$-well-ordered (see Chapter 9) then it is well-founded, but a well-founded set is not necessarily $\in$-well-ordered. If a set A is well-founded, this fact has several far-reaching consequences that are very easy to prove:

11.5 Theorem Let A be a well-founded set. Then the following are true in A:

a) For every $x \in A$, $x \notin x$.

b) For any $x, y \in A$, if $x \notin y$ then $y \notin x$.
c) In A, there is no infinite descending set of elements $\ldots \in x_3 \in x_2 \in x_1 \in x_0$.
d) A is well-founded iff $\exists x \in A$ such that $x \cap A = \varnothing$.

*Proof.*For (c), the set $\{x_i : i \in \omega\}$, if it existed, would have no minimal element.

(a) If there were an $x \in A$ such that $x \in x$, we'd have a descending sequence $\{\ldots x \in x \in x\}$.
(b) Here we would have a descending sequence $\{\ldots y \in x \in y \in x\}$. Finally, the proof of (d) is left as an exercise. ■

It was mentioned above that in axiomatic set theory, a set is whatever can be built up from the empty set by iterating the power set and union steps. That's the point of the concept of well-founded. The set membership relation can go down only a finite number of times, thus proving that every possible set is at some stage in a hierarchy of set membership.

11.6 DefinitionA set A is said to be *transitive* if every element of A is a subset of A. Equivalently: $(x \in A) \wedge (y \in x) \Rightarrow (y \in A)$.

We have seen (Chapter 9) that every ordinal number is a transitive set.

From Parts (a) and (b) of the last theorem, if a set A is transitive and well-founded, this means that $\in$ is an order relation on A. (Specifically, it is a strict order relation $<$.) However, $\in$ is not a linear order on A, because two arbitrary elements $x, y \in A$ are not necessarily comparable. In fact, if $\in$ were a linear order on A, then a well-founded set A would be an $\in$-well-ordered set, and therefore A would be an ordinal.

11.7 Lemma

a) Every subset of a transitive set is transitive.
b) Every union of transitive sets is transitive.
c) The power set of a transitive set is transitive.

(The simple proofs are left as exercises.)

11.8 Theorem For any two ordinal numbers ξ and α,

a) V_α is a transitive set, and
b) $\xi < \alpha \Rightarrow V_\xi \subset V_\alpha$.

*Proof.*We prove (a) and (b) jointly by induction on α.
Assume (a) and (b) are true for all $\beta < \alpha$. The claims are trivial for $\alpha = 0$. If α is a limit ordinal, then (a) and (b) follow immediately from Lemma 11.7(b). If α is a successor ordinal, $\alpha = \beta + 1$, then $V_\alpha = \mathscr{P}(V_\beta)$. Since V_β is transitive, it follows from Lemma 11.7(c) that V_α is transitive. Finally, $V_\beta \subset V_\alpha$ from the definition of the cumulative hierarchy. ■

The sets V_α have a property that is, more or less, a converse of transitivity:

11.9 Lemma If $x \subset V_\alpha$ then $x \in V_{\alpha+1}$.

*Proof.*From the definition of the cumulative hierarchy, the elements of $V_{\alpha+1}$ are the subsets of V_α. ■

To reach our goal, we must prove now that every V_α is well-founded:

11.10 Theorem For every $\alpha \in OR$, V_α is well-founded.

*Proof.*We begin by showing the following: $(x \in V_\alpha) \wedge (y \in x) \Rightarrow y \in V_\beta$ for some $\beta < \alpha$. The proof is by induction on α. If α is a limit ordinal then $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$. Then $x \in V_\alpha \Rightarrow x \in V_\beta$ for some $\beta < \alpha$. By the hypothesis of induction, $y \in V_\beta \subset V_\alpha$. Next, if $\alpha = \beta + 1$, then $V_\alpha = \mathscr{P}(V_\beta)$. So if $x \in V_\alpha$ then $x \subset V_\beta$,so $y \in x \Rightarrow y \in V_\beta$. Now for our main result:

Let Y be a non-empty subset of V_α. Let β be the least ordinal such that $Y \cap V_\beta \neq \varnothing$. Obviously $\beta < \alpha$, since $Y \cap V_\alpha = Y$. Let x be any element of $Y \cap V_\beta$. By the previous paragraph, $y \in x \Rightarrow y \in V_\delta$ for some $\delta < \beta$. But β was the least ordinal for which $Y \cap V_\beta \neq \varnothing$, hence $y \notin Y$, which is a contradiction. Thus, x is $\in$-minimal. ■

We have shown that every V_α is transitive and well-founded. Can the same be said for sets that are *members* of V_α? Let's try:

11.11 Theorem If $A \in V_\alpha$, then A is well-founded.

*Proof.*If X is any non-empty subset of A, we wish to prove that X has an $\in$-minimal element. Well, let α be the minimal rank among all elements in X, and let $y \in X$ be an element whose rank is α, hence $y \in V_\alpha$. Now suppose there is an element $x \in X$ such that $x \in y$: Then $x \in y \Rightarrow x \in V_\alpha$ because V_α is transitive. But that is impossible because $x \in y$ implies that the rank of x must be less than the rank of y which is minimal in X. So x cannot exist, hence y is minimal in X. ■

Next, it would be nice if we were able to show that for every ordinal α, every member of V_α is a transitive set. If that were the case, we could characterize V as the collection of all transitive well-founded sets. Unfortunately, we cannot do this because it turns out that not every set in V is transitive. When mathematicians find a roadblock of this kind, they usually search for a detour that leads to the same result. The roadblock we have just encountered motivates the following definition:

11.12 Definition The *transitive closure* of a set A is the smallest transitive set that contains A.

The transitive closure of a set A is easy to construct by using the Axiom of Union. Begin with $\bigcup_1(A) = \bigcup A, \bigcup_2(A) = \bigcup\bigcup_1(A)$ and for every n, $\bigcup_{n+1}(A) = \bigcup\bigcup_n(A)$. Then the transitive closure of A is defined to be the set

$$tc(A) = A \cup \bigcup_1(A) \cup \bigcup_2(A) \cup \bigcup_3(A) \cup \cdots$$

Intuitively, $tc(A)$ is the set of all objects which are elements of elements of…of elements of A (iterated a finite number of times). It is a simple exercise to verify that $tc(A)$ is a transitive set that contains A, and moreover, that if B is a transitive set such that $A \subseteq B$, then $tc(A) \subseteq B$.

11.13 Lemma For any ordinal α, $A \in V_\alpha$ iff $tc(A) \in V_\alpha$.

Proof. From the previous paragraph: $tc(A)$ is a transitive set that contains A, and moreover, if B is a transitive set such that $A \subseteq B$, then $tc(A) \subseteq B$. Because V_α is transitive, if $A \in V_\alpha$ then $A \subseteq V_\alpha$. Thus, (with V_α in the role of B above), $tc(A) \subseteq V_\alpha$. Conversely, if $tc(A) \in V_\alpha$, then $tc(A) \subseteq V_\alpha$ so $A \subseteq tc(A) \subseteq V_\alpha$. So from Lemma 11.9, $A \in V_{\alpha+1}$. ■

In particular, if A is a member of some V_α, then $tc(A) \in V_\alpha$ and by Theorem 11.11, $tc(A)$ is well-founded. Thus every member of V has a transitive well-founded closure. In our last theorem of this section we shall prove the converse of the above, namely: Every set whose transitive closure is well-founded is a member of V. It will follow from this that we have fully characterized V: It is the class of all the sets whose transitive closure is well-founded.

Compare this with a corresponding result about ordinal numbers in Chapter 9, Section 5: OR (the class of all ordinal numbers) is the class of all transitive ∈-well-ordered sets. Since every ∈-well-ordered set is well-founded—and since every transitive set is its own transitive closure—it follows that *every ordinal number is a member of V*.

11.14 Theorem Every transitive well-founded set is a member of V.

*Proof.*Let A be a transitive, well-founded set. It suffices to prove that $A \subseteq V$, for by Lemma 11.9 it follows that $A \in V$. If $A \nsubseteq V$, let y be a minimal element of $A - V$, and let $z \in y$. Then $z \in A$ because A is transitive. We have just shown that $y \subseteq A$. By Lemma 11.9 $y \in V$. But this contradicts $y \in A - V$, and from this contradiction we conclude that $A \subseteq V$. ■

The central fact about V—the reason for its importance in mathematics—is that all the axioms of set theory hold when the word “set” is replaced by the words “transitive well-founded set”. That is the reason why V is called the *universe of sets*. The class V contains all the ordinal numbers (hence the cardinals as well), and all the constructions needed in mathematics can be carried out on the members of V.

In V, sets are built from the bottom up. The members of V are clear, clean, well defined objects which satisfy simple axioms, so set theory on V is a branch of mathematics as rigorous as other areas such as algebra and analysis. Notice that we have stated that all the axioms of set theory apply to V. The proofs are straightforward verifications, and several of them are given in the exercises at the end of this section. It is therefore perfectly reasonable to assert that all the sets that exist in mathematics are members of V. If you wish to consider the members of V to be all the sets there are, then you must legislate this fact as an axiom:

Axiom of FoundationV is the class of all sets, that is, $V = \mathscr{U}$.

From this point onward, we take the Axiom of Foundation to be one of our axioms.

EXERCISES 11.1

1. Suppose that in a set A there is no descending set of elements $\ldots \in x_2 \in x_1 \in x_0 \in A$. Prove that A is well-founded.
2. Prove that a set A is well-founded iff $\exists x \in A$ such that $x \cap A = \varnothing$.

3. Prove that every subset of a well-founded set is well-founded, and every element of a well-founded set is well-founded.
4. Prove that if x and y are well-founded, so are $\{x\}, \{x, y\}, \bigcup x$ and $\mathscr{P}(x)$.
5. Prove that every subset of a transitive set is transitive.
6. Prove that any union of transitive sets is transitive.
7. Prove that the power set of any transitive set is transitive.
8. Prove that a set A is transitive iff $\bigcup A \subseteq A$.
9. Prove that a set A is transitive iff $A \subseteq \mathscr{P}(A)$.
10. Suppose that α is a limit ordinal. Prove (a) If $\xi < \alpha$ then $V_\xi \subseteq V_\alpha$.

    (b) If V_ξ is transitive for every $\xi < \alpha$ then V_α is transitive.
11. If A is a set and $tc(A)$ is its transitive closure, prove: (a) $tc(A)$ is transitive.

    (b) $A \subseteq tc(A)$.
12. Prove: If A is a set, B is a transitive set, and $A \subseteq B$, then $tc(A) \subseteq B$.
13. Prove (a) If A is a transitive set then $A = tc(A)$. (b) If $x \in A$ then $tc(x) \subseteq tc(A)$.
14. If $A \in V$ and $B \subseteq A$ then $B \in V$. (You may use Lemma 11.9).
15. Prove: if $x, y \in V$, then $\{x\}$, $\{x, y\}$,$x \times y$, and $x \cap y$ are in V.
16. Prove: For all $n \in \omega$, $|V_n|$ is finite. (Prove by induction on n.)
17. Every well-founded set is in V. Conclude that V is the class of well-founded sets.
18. The axiom of pairs states that if x and y are sets, then $\{x, y\}$ is a set. To prove the axiom of pairs for V is to show that if $x, y \in V$ then $\{x, y\} \in V$. Prove this, using the rule for constructing the cumulative hierarchy.
19. The axiom of union asserts that if A is a set, then $\bigcup A = \bigcup\{x : x \in A\}$ is a set. To prove the axiom of union on V is to show that if $A \in V$ then $\bigcup A \in V$. Prove this.
20. The power set axiom states that if A is a set, so is $\mathscr{P}(A)$. Prove that the power set axiom holds in V. That is, if $A \in V$, then $\mathscr{P}(A) \in V$.

2 MODELS

Whenever mathematics is done rigorously, one begins with a formal language $\mathscr{L}$ which has symbols to denote the relations and operations pertaining to the mathematical theory under discussion. For example, when studying ordered sets, one uses a language $\mathscr{L}$ with two binary relation symbols < and =, together with the logical connectives ∧, ∨, ¬ and the quantifiers ∃ and ∀. For the theory of rings we use a language $\mathscr{L}'$ with a binary relation symbol =, operation symbols ·, + and −, and the logical connectives and quantifiers.

If $\mathscr{L}$ is a formal language, one obtains a theory T in the language $\mathscr{L}$ by laying down a set of axioms for the theory. For example, if T is the theory of partially ordered sets, its axioms are:

$$\forall x, [x \leq x],$$
$$\forall x, y, [(x \leq y) \wedge (y \leq x) \Rightarrow (x = y)]$$
$$\forall x, y, [(x \leq y) \wedge (y \leq z) \Rightarrow (x \leq z)]$$

Finally, a *model* of a theory *T* is a set *A* together with a relation on *A* corresponding to each relation *symbol*, and an operation on *A* corresponding to each operation symbol. For example, a model for the theory of ordered sets would consist of a set *A* together with a relation on *A* (*that is, a set of ordered pairs of A*) satisfying the three axioms above. In plain language, a model for the theory of ordered sets is an ordered set. Likewise, a model for the theory of rings is a ring.

A mathematical theory is said to be *consistent* if it does not harbor any contradictions, that is, you cannot use the axioms of the theory to prove a statement *F* as well as its negation ¬*F*. Proving that a theory *T* is consistent is notoriously difficult, because in the worst case, you would have to see every theorem of *T* to know whether or not there are contradictions. However, one of the first results of model theory is called the Completeness Theorem and states that a theory *T* is consistent if and only if it has a model. In practice, this is the easiest—and sometimes the only—way to prove that a theory is consistent. In plain language, it is only if you produce a model of a theory that you can be certain it is consistent.

One of the most productive branches of mathematics during the 20th Century (it occupied mathematical geniuses such as Gödel and von Neumann during much of their careers) has been the study of models of set theory. Note that set theory cannot be taken seriously as a rigorous discipline in mathematics until it is shown that there are models of set theory, because it is only once we see a model of the axioms of set theory that we know it is consistent.

However, a model of set theory is not like any other model: For example, a model of group theory is a set with an operation that satisfies the usual group axioms. In the same way, a model of set theory would have to be a set whose members (sets) satisfy the axioms of set theory. Achieving such a thing is fraught with difficulties. In the first place, you want a set to serve as a model of set theory, but it is only inside set theory that you come to know what a set is. Secondly, if it is a model of set theory we are constructing, then—in that model—we are able to construct every ordinal number, and we know that the class of all the ordinal numbers is a proper class: So the set, in our model, would have to contain a proper class, which is impossible.

At the very time when these questions were being asked, a remarkable result about models—called the Löwenheim-Skolem Theorem—was discovered. It was proved that *every consistent theory has a countable model*. In other words, if a theory has a model of any size, it also has a countable model. So if set theory is consistent, not only must it have a model, but it must have a *countable* model. Imagine: a countable model for all of set theory! In the 1920s, ingenious models were constructed for subsets of the axioms, but never for the full set of axioms of set theory. This quest was abandoned at the beginning of the 1930s when Kurt Gödel dropped a bombshell on the mathematical world.

3 INDEPENDENCE RESULTS IN SET THEORY

In 1931, as part of his doctoral dissertation, Kurt Gödel proved that if a consistent system of axioms is sufficient to prove the theorems of arithmetic, then the consistency of such a system cannot be proved by the standard methods of mathematics. As we have seen in this book, it is possible to construct the natural numbers and carry out arithmetic within axiomatic set theory. Thus, by Gödel’s Theorem, if axiomatic set theory is consistent, then its consistency cannot be proved. In particular, we must abandon our quest to construct a model of set theory.

Where does this leave us? We appear to have just two choices: Either we choose to accept the consistency of axiomatic set theory on faith, or alternatively, we abandon the ideal of founding mathematics on axioms and rigorous proof, and put our faith in intuition instead. Most mathematicians accept the consistency of the various axiomatic systems of set theory because the axioms strike us as absolutely plausible and almost undeniably true. Primarily, they are willing to accept the axioms of one system of set theory (axioms similar to the ones we have been using) called the Zermelo-Fraenkel axioms and discussed in the next Section. This system, known at ZF set theory, does not require proper classes because it places other limitations on what is allowed to be a set.

I have said that the axioms of set theory seem perfectly plausible. That is true with two notable exceptions: The Axiom of Choice and the Continuum Hypothesis. We like them because they simplify and beautify mathematics, but they do not strike us as unquestionably true. Thus, for many years the Holy Grail after which the most gifted knights of mathematics quested was a proof of the axiom of choice (AC) and the continuum hypothesis (CH) from the other axioms of set theory. Once again, it was Kurt Gödel who settled the issue in a most provocative way: In 1936 he published a proof that both AC and CH are *consistent* with the other axioms of set theory. Specifically, he showed that—assuming the other axioms of set theory are consistent—then ¬AC and ¬CH cannot be deduced from them.

This discovery set off a new race to prove that the negations of AC and CH (denoted by ¬AC and ¬CH) are likewise consistent with the other axioms. It was not until thirty four years later that this result was achieved, using very new ideas, by Paul Cohen. Taken together, what these results show is that *AC and CH are independent of other axioms of set theory*. If you want AC in set theory, you must state it as an independent axiom, and if you want CH then you must state CH as an independent axiom. And for that matter, you may add ¬AC as a new axiom without any danger of contradiction, and likewise you may add ¬CH. The proofs of these results are called the Independence Proofs of set theory.

There was a time when people asked: “After all, is the Continuum Hypothesis true or not? Is the Axiom of Choice true in reality or is it not?” We no longer ask these questions today because we have become a little more cynical about absolute truth. Few people today believe in the Platonic heaven where all truths are enthroned forever. Most mathematicians don’t ask such questions, but generally accept AC and CH because it makes their work easier. For the Philosophical Few, it is clear that mathematics is a great fountain of abstract ideas, perfect if viewed from within, somewhat flawed if viewed from without. In an axiomatic system you lay down a set of axioms and work to deduce the consequences of your axioms. You do not ask if your axioms are “true”.

A perfect exemplar for this is the Parallel Postulate in Euclidean geometry. For two thousand years brilliant thinkers burned out their synapses in vain attempts to prove the Parallel Postulate from the other axioms of geometry. One fine day people realized that there are other kinds of geometry in which the Parallel Postulate does not hold in its Euclidean form. It turns out, then, that the Parallel Postulate is neither true nor false. That is the very situation we have today with the Axiom of Choice and the Continuum Hypothesis. It seems very likely that there will be useful applications in mathematics for ¬AC and ¬CH.

4 THE QUESTION OF MODELS OF SET THEORY

We have seen, above, that axiomatic set theory is consistent if and only if it has a model. The foremost problem with building a model of set theory is that it would be too big. For example, V is a class of sets that satisfies all the axioms of set theory, but it is too large to be a model: V is a proper class, whereas a model must be a set. The only way out of this trap is to ingeniously construct a model *M* whose elements *mimic* all the properties of sets, but such that *M* does not truly contain every set. In fact, in view of the Löwenheim-Skolem theorem, the model *M* we construct may be countable.

The idea for achieving this is deceptively simple: We mimic the construction of the cumulative hierarchy $\{V_\alpha\}_{\alpha\in ON}$ while placing restrictions on the sets that are admitted into each V_α. This is done in such a way as to keep only those sets that are indispensable to our arguments, while excluding all the “inconvenient” sets. The resulting universe of sets *M* is a set—and we recall that “sets” include such things as functions, relations, the natural numbers, and so on.

As a model of set theory, *M* must include an uncountable set *y* (in fact many uncountable sets). A set *y* is uncountable if there is no 1-1 correspondence between *y* and *ω*. Remember that *M* does not contain *all* sets—many "true" sets are absent from *M*—and among the sets absent from *M* are all the bijective functions between *ω* and *y*. As a result, in the model *M* there is no 1-1 correspondence between *ω* and *y*, and that makes *y* uncountable by definition. This is true even though from *outside* the model, *y* is countable—perhaps even finite. The idea, then, is that there are two parallel universes: The universe of sets as observed from *inside* the model *M*, and the universe of "true" sets as observed from outside the model.

For example, in *M* let *x* be the set that plays the role of $\aleph_0$ and *y* the set that plays the role of $\mathscr{P}(\aleph_0)$. Then *y* does not really contain every subset of $\aleph_0$, it contains only those subsets of *x* that are in *M*. So if *M* is a countable model then *y* must be countable, and so clearly *y* cannot be equal to $\mathscr{P}(\aleph_0)$ for the outside observer. The set *y* that we call the powerset of $\aleph_0$ in *M* is not the same as the "real" powerset of $\aleph_0$ because many subsets of $\aleph_0$ are missing in *M*.

What we have just related is the underlying idea behind Gödel's proof—but the devil is in the details. A full account of Gödel's results and those of Paul Cohen is beyond the scope of this book. But I shall outline the principal stages of the argument. The first problem is the construction of a model *M* of set theory having the properties described in the previous paragraphs. The key idea is that of *constructible sets*.

In a true universe of sets, if A is any set then by the axioms of set theory, the universe must contain such things as $\mathscr{P}(A), \bigcup A$ and all the combinations of *A* with other sets in the universe. So the supply of sets rapidly outstrips the boundaries of a countable universe, or any universe which is a set. What must be done at the very start, then, is to restrict what counts as a *set*. The idea is that the only kinds of set that can play any role in set-theoretic proofs are sets that can be described in formulas of the language of set theory. It stands to reason that indescribable sets can play no part in any reasoning about sets.

Recall that an expression in the formal language of set theory is any formula that can be written as a valid combination of the logical connectives ∨, ∧, ¬, the quantifiers ∀ and ∃, and the relation ∈. A formula may also contain parameters, that is, constant symbols referring to objects already constructed at an earlier stage of a proof. An arbitrary formula with free variables $x_1, x_2, \ldots, x_n$ may be written as $\phi(x_1, x_2, \ldots, x_n)$ without displaying the parameters.

This is a timely moment to say a word about the Zermelo-Fraenkel (ZF) axioms of set theory, and how they differ from the axioms we have been using throughout this book. The distinction boils down to an important difference between the Axiom of Class Construction (Axiom A2) used in this book, and the corresponding axiom in the ZF system, called the Axiom of Selection. The Axiom of Class Construction asserts the existence of a class *C* consisting of all sets *x* that satisfy a formula $\phi(x)$. In many cases, the class *C* formed in this manner is a proper class. The Axiom of Selection in the ZF system is more conservative: It asserts that if *A* is any set and $\phi(x)$ is a formula, there is a *set S* consisting of all *x* such that $x \in A$ and $\phi(x)$. It asserts the existence of sets that are subsets of existing sets, hence it cannot give rise to proper classes.

In ZF set theory, if *A* and *B* are sets, then *A* is said to be *constructible over B* if there exists a formula $\phi(x)$ in the formal language of set theory such that *A* consists of all the elements of *B* which satisfy $\phi(x)$. The formula ϕ may contain parameters (constant symbols) which refer to elements of *B*. That is, we may form:

$$A = \{x \in B : \phi(x) \text{ and the parameters in } \phi \text{ refer to elements of B}\}$$

We now revise the cumulative hierarchy by restricting sets to constructible sets. The *constructible* *universe* is built as a hierarchy of sets indexed by the ordinals, just as the cumulative hierarchy was. In each successor step, instead of adding all subsets of the current set, only the definable ones are added. That is, we replace the power set operation by the *constructible power set* operation defined as follows:

$$\mathscr{P}^*(A) = \{X \subseteq A : X \text{ is a constructible set }\}$$

This small difference makes a big difference: The reason is that in a countable formal language there are only countably many formulas, hence there are only countably many constructible sets. Thus, even if *A* is an infinite set, $\mathscr{P}^*(A)$ has no more than countably many members. Except for this one difference, the constructible hierarchy is defined like the cumulative hierarchy.

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= \mathscr{P}^*(L_\alpha) \\ L_\alpha &= \bigcup\nolimits_{\beta<\alpha} L_\beta \quad \text{if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Finally, *L* is defined to be the union of all the levels: $L = \bigcup_{\alpha \in OR} L_\alpha$

L is called the *constructible universe*. Now let's slow down a moment, because a reader sitting in the back of the room has a question:

—"Author: You tell us on the one hand that by Gödel's incompleteness theorem, it is impossible to prove the consistency of the ZF axioms. You also tell us that consistency means that there exists a model. And finally, you speak to us about constructing models of set theory. Surely if you construct such a model, you have proved the consistency of ZF which you say is impossible. What gives?"

—Thank you, that's an excellent question! The model we want to construct is built *within* ZF set theory. So if ZF happens to be inconsistent to begin with, then the model we have built is flawed, since it is built within the constraints of ZF. So our model is only a model of ZF *on the prior assumption that ZF is consistent*. Consequently, any result we are able to derive from this model is conditional on the consistency of the ZF axioms.

EXERCISES 11.4

1. a) Write the definition of $x = y$ as a formula ϕ in the language of set theory. (See Definition 1.9 for the formal definition of the equality of sets.)

   b) Write a formula ψ which states Axiom A1 in the language of set theory.

   c) The set $\{a, b\}$ is the set $\{x : \phi(x)\}$. Write the correct formula for $\phi(x)$.

   d) The set $\bigcup A = \{x : \psi(x)\}$. Write the correct formula for $\psi(x)$.

   e) The set $\mathscr{P}(A) = \{x : \xi(x)\}$. Write the formula for $\xi(x)$.

   f) The ordered pair $(a, b) = \{x : \chi(x)\}$. Write the formula for $\chi(x)$. (Harder than (a)–(e).)

2. If *A* is a class and ϕ is a formula in the language of set theory then ϕ^A is the formula obtained by replacing every quantifier $\exists x$ in ϕ by $(\exists x \in A)$ and every $\forall x$ by $(\forall x \in A)$.

   a) Prove that if $B \subseteq A$, then ϕ is true in *B* iff ϕ^B is true in A.

   b) Give an informal example (e.g. using finite sets) showing that there are classes $B \subseteq A$ and formulas ϕ such that ϕ^B is true in A but ϕ is not true in A, and where ϕ is true in *A* but ϕ^B is not true in A.

3. Suppose a and b are sets such that $a, b \in L$.
   a) Prove that $\{a, b\} \in L$, and explain from this why the axiom of pairs is true in L.
   b) Prove that the ordered pair $(a, b) \in L$.
   c) Prove that $\bigcup a \in L$. Explain why the axiom of union is true in L.
4. Each of the following is a set $\{x : \phi(x)\}$ where ϕ may contain parameters. For each of the following sets, give the formula ϕ in the language of set theory, and indicate which are the parameters in ϕ. Explain why each is—or is not—a constructible set.
   a) $a \cup b$.
   b) $a \times b$.
   c) (a, b), where (a, b) is an ordered pair.
   d) $\mathscr{P}(a)$.

5 PROPERTIES OF THE CONSTRUCTIBLE UNIVERSE

The hierarchy of constructible sets has many of the same properties as the cumulative hierarchy, but the style of reasoning about constructible sets is more subtle than the way we reason about conventional sets. Look carefully at the proof of the next theorem.

11.15 Theorem For each ordinal α, the following are true:

a) L_α is transitive.
b) For $\beta < \alpha$, $L_\beta \subseteq L_\alpha$.
c) For $\beta < \alpha$, $L_\beta \in L_\alpha$.

*Proof.*The first two statements are proved jointly by induction on α. Thus, suppose (a) and (b) are true for β. In particular, L_β is transitive. If $\alpha = \beta + 1$, then $L_\alpha = \mathscr{P}^*(L_\beta)$. If $x \in L_\beta$, let $A = \{a \in L_\beta : a \in x\}$. Note that A is the set of all elements $a \in L_\beta$ that satisfy the formula $a \in x$, where x is a parameter that refers to an element of L_β. (You are free to replace the parameter x by an n-tuple $x_1, x_2, \ldots x_n$ of parameters.) Thus, A is a constructible set. Moreover, A consists of all the elements of L_β that are elements of x: In other words $A = x$. Thus, $x \in L_{\beta+1} = L_\alpha$, which shows that $L_\beta \subseteq L_\alpha$.

Moreover, if $x \in L_\alpha$ then $x \subseteq L_\beta$, hence $x \subseteq L_\alpha$ from the previous line. Thus, L_α is transitive. Consequently, (a) and (b) are true for α if α is a successor ordinal. If α happens to be a limit ordinal, the result is very simple.

Now for Part (c): Note that the formula $x = x$ is trivially a formula (without parameters) in the language of set theory, hence $\{a \in L_\beta : a = a\} = L_\beta$ is an element of $L_{\beta+1}$. This proves Part (c) if α is a successor ordinal, and the proof is immediate if α is a limit ordinal. ■

11.16 CorollaryL is transitive.

*Proof.*Indeed, if $x \in L$ then $x \in L_\alpha$ for some ordinal α. Since L_α is transitive, it follows that $x \subseteq L_\alpha \subseteq L$. ■

Two more simple facts about L will prove useful, and are stated as a lemma:

11.17 Lemma For every ordinal α,

a) $\alpha \in L_\alpha$.
b) $L_\alpha \subseteq V_\alpha$.

Proof.(a) is an easy induction on α: Suppose our claim is true for L_α, and prove it for $L_{\alpha+1}$. Then $\alpha \subseteq L_\alpha$ because L_α is transitive. Thus, $\alpha + 1 = \alpha \cup \{\alpha\} \subseteq L_\alpha$ hence $\alpha + 1 \in L_{\alpha+1}$. The case where α is a limit ordinal is similar, but simpler. As for (b), we note that $\mathscr{P}^*(L_\alpha) \subseteq \mathscr{P}(\alpha)$, hence $L_{\alpha+1} \subseteq V_{\alpha+1}$. ■

From Lemma 11.17 (a) it follows that L *contains all the ordinal numbers*. L is transitive as shown above, and as we are about to see, L satisfies all the ZF axioms. Any transitive class M that contains all the ordinals and satisfies the ZF axioms is called an *inner model* of set theory.

As mentioned earlier, there is a strong intuitive basis for considering L to be the class of all sets. By definition, L contains all the sets that are describable by a formula in the language of set theory. And there is no practical reason to admit sets which lack any description, for we would never make use of such sets. They would merely sit there and muddy the waters. Thus, from this point onward we shall assume the following important axiom:

Axiom of Constructibility Every set is constructible, that is, every set is in L. This axiom is usually denoted by the symbol V = L.

As you have just seen, in order to prove statements about a constructible set A, it is necessary to keep in mind that A consists of all the elements x (in some set) that satisfy a formula $\phi(x)$. A crucial subtlety is that an assertion $\phi(x)$ may hold in a set A but may fail to hold in a subset $B \subset A$, or vice-versa. For example, let $A = \mathscr{P}\{a, b, c, d, e\}$, and let $B = \{\{a\}, \{a, b\}, \{a, b, c\}\}$. Then B is strictly ordered by $\subset$ (hence B satisfies a formula which states that it is strictly ordered) but A does not because A is not strictly ordered by $\subset$. To be precise, B satisfies the formula $\phi = (\forall x, y)[x = y \vee x \subset y \vee y \subset x]$ but A does not satisfy that formula. However, A satisfies $\phi^B = (\forall x, y \in B)[x = y \vee x \subset y \vee y \subset x]$. This latter formula, denoted by ϕ^B, is called the formula ϕ *relativized to B in A*. It is obvious that ϕ does not hold in A,but ϕ^B does. You may think of ϕ^B as a voice in A that makes a statement about B.

The notion of relativized formulas is clearly essential for reasoning about constructible sets and their subsets. We now give a precise definition for it:

11.18 Definition If A is any class and ϕ is a formula in the language of set theory then ϕ^A, called the *relativization* of ϕ to A, is the formula obtained by replacing every quantifier $\exists x$ in ϕ by $(\exists x \in A)$, and likewise replacing every quantifier $\forall x$ by $(\forall x \in A)$.

It is clear that if A and B are classes such that $B \subseteq A$, then ϕ is true in B iff ϕ^B is true in A. However, if ϕ^B is true in A this does not entail that ϕ is true in A. This fact is illustrated in the previous example, and motivates the definition that follows.

11.19 Definition Let $A \subseteq V$. We say that ϕ is *absolute for A in V* if for all $x_1, \ldots, x_n \in A$,

$$V \vDash \phi(x_1, \ldots, x_n) \text{ iff } V \vDash \phi^A(x_1, \ldots, x_n).$$

Informally, ϕ and ϕ^A are equivalent in V if all the free variables of ϕ take values in A. We say that ϕ is *absolute* if it is absolute for every A.

If all the important formulas were absolute, it would make life simpler for set theorists. Mainly, it would be possible to form “small” models of set theory. Unfortunately, that is not the case. As the next example shows, even a simple formula such as $x \subseteq y$ is not absolute.

ExampleLet $A \subseteq V$ be the set whose only elements are $\varnothing$ and $\{\{\varnothing\}\}$. If ϕ^A is the formula $(\forall x \in A)(x \in \{\{\varnothing\}\} \Rightarrow x \in \varnothing)$—equivalently $\{\{\varnothing\}\} \subseteq \varnothing$—then ϕ is true in V. [Note that the only $x \in \{\{\varnothing\}\}$ is $\{\varnothing\}$,but $\{\varnothing\}$ is not in A]. However, ϕ is not true in V.

11.20 DefinitionA formula ϕ is said to be *absolute over transitive domains* if, for every transitive class A, ϕ is absolute for A in V.

The good news is that a great many formulas are absolute over *transitive domains*. All the models that we shall be working with from this point onward will have transitive domains. To begin, it will be shown that Axiom A1 is absolute over transitive domains. That is, Axiom A1 is true in every transitive class. Please note carefully how the proof works.

11.21 Lemma If A is a transitive class, then A satisfies Axiom A1. (In other words, the formula for Axiom A1 is absolute for transitive classes.)

*Proof.*Axiom A1 is the following formula $\phi : (\forall x, y)[(x = y) \Leftrightarrow ((\forall u)(u \in x \Leftrightarrow u \in y)]$.

Then ϕ^A is: $(\forall x, y \in A)[(x = y) \Leftrightarrow ((\forall u \in A)(u \in x \Leftrightarrow u \in y)]$. It is obvious that if ϕ is true in V, then *a fortiori* ϕ^A is true. For the converse, suppose ϕ^A holds in V, and show that ϕ holds in V for elements $x, y \in A$. Our proof is by contradiction: We shall assume there is a u such that $u \in x$ and $u \notin y$. Since A is transitive, $u \in A$. Since we assume that ϕ^A holds, it follows from $u \in x$ that $u \in y$, and this is a contradiction. Thus, ϕ is true in A. ■

At the heart of the proof is the fact that—because A is transitive—every element of x is in A, and every element of y is in A. Note that the argument would not work if A were not transitive. Many other properties similarly illustrate the importance of transitivity. For instance, look at the notion of function, and suppose f is a function in V: Then f is a set of ordered pairs $(x, y) = \{\{x\}, \{x, y\}\}$. If A is not transitive, we cannot prove that x is in A or y is in A, hence we cannot prove that f is a function in A. (But if A *is* transitive, we can.)

Many other properties and relations are absolute for transitive models. These include: being an ordered pair, a function, a 1-1 function, a relation, the domain or range of a relation, the set $x \times y$, being an ordinal number, the set ω, and many others. In fact, the next theorem yields a treasure-trove of formulas that are absolute on transitive domains.

11.22 DefinitionA formula ϕ is called a Δ_0 formula if all of its quantifiers are bounded. That is, all quantifiers in ϕ are of the form $\exists x \in y$ or $\forall x \in y$ for a set y.

11.23 Theorem If A is transitive and ϕ is a Δ_0 formula, then ϕ is absolute for A.

*Proof.*It is clear that $x = y$ and $x \in y$ are absolute for any A, because they do not change when you relativize them. In fact, all quantifier-free formulas ϕ are unchanged when relativized, that is, $\phi = \phi^A$. The proof of the theorem is by induction on the length of ϕ, which is defined to be the number of occurrences of logical operators $(\wedge, \vee, \neg, \exists, \forall)$ in ϕ. We have already given the proof for formulas of length 0 (namely $x \in y$ and $x = y$).

Now, if ϕ and ψ are absolute for A, then clearly, so are $\neg\phi$, $\phi \wedge \psi$ and $\phi \vee \psi$. The delicate step of the proof begins here: Assume that ϕ is of the form $(\exists x \in y)\psi$ and suppose that ϕ^A is true in V, where y and other possible free variables in ψ represent elements in A. Since A is transitive, $x \in A$, and therefore $(\exists x \in A)[x \in y \wedge \psi^A]$ holds in V. By the hypothesis of induction, ψ holds iff ψ^A does, hence $(\exists x \in A)\ (x \in y \wedge \psi) = (\exists x \in y)(\psi) = \phi$ holds in V.

Conversely, suppose ϕ is true in V, that is, $(\exists x \in y)\psi$ is true in V, where y and other possible free variables in ψ represent elements in A. Since A is transitive, $y \in A$. Now, $\phi^A = (\exists x \in y)\psi^A$, and the proof is essentially the same as in the previous paragraph. ■

Other broad categories of formulas can also be proved to be absolute over transitive domains. Actually, what is even more important in mathematics is the fact that many simple properties *fail* to be absolute for transitive models. The reason this is desirable is that we wish to construct models M having properties that hold in M but are not consequences of ZF. For example, we would like to find a transitive model M in which the negation of the axiom of choice (¬AC) holds. This would imply that ZF (without AC) does not imply AC. The very purpose of the endeavor of creating models of set theory is to find models having properties that are not absolute. The basis of Gödel's independence results is that the concept of a well-ordering, as well as the concept of cardinality, are not absolute.

Our mission now is to prove that *if L is a model of ZF*, then a number of additional properties also hold in *L*. And because these properties hold in *L*, they must be consistent with ZF. To be explicit, if a property ϕ is true in *L* (which is a model of ZF) then it is not possible to prove $\neg\phi$ in ZF, because we are assuming that ZF is consistent on account of its having a model.

11.24 *Remark.* In order to prove that a formula ϕ holds in the model *L*, what we need to show is that $ZF \vdash \phi^L$. The reason is that we are assuming (this is merely an assumption hence requires no proof) that *L* is a model of ZF. So if $ZF \vdash \phi^L$ then ϕ^L is true in every model of ZF, hence in *L*. Thus, ϕ^L, and therefore ϕ itself, are true in the model *L*.

From the previous Remark, in order to show that the axioms in ZF hold in *L*, we must show that for each axiom ϕ, the formua ϕ^L is true. It is this task that we undertake next. We are substantially aided by the fact that *L* is a transitive class.

11.25 Theorem For every axiom ϕ of ZF, ϕ^L is true in *L*.

Proof. To carry out this proof, it is essential first, to be clear about the relationship between an axiom ϕ of ZF and the corresponding formula ϕ^L. We have already seen that in V, the unordered pair $\{a, b\}$ is the set $\{x : x = a \vee x = b\}$. Note that the set is defined by the formula $\phi = (x = a \vee x = b)$. But in L, $\{a, b\}$ is the set $\{x \in L : x = a \vee x = b\}$. You will note that ϕ is the same as ϕ^L, so the operation of forming ordered pairs is absolute.

For unions, suppose that $a \in L$: Then $\bigcup a$ is the set of all x that satisfy the formula $\phi = (\exists y \in a)(x \in y)$. So relativized to L, $\bigcup a = \{x \in L : (\exists y \in a)(x \in y)\}$. Note the importance of assuming that L is transitive: If $a \in L$ and $y \in a$ then you conclude that $y \in L$. As you can see, in this case ϕ^L is the same formula as ϕ, hence ϕ is absolute over L. Since we have shown that unions are the same in L as in V, the Axiom of Unions holds in L.

Likewise, for subsets, $z \subseteq x$ is an abbreviation for $\phi = (\forall v)(v \in z \Rightarrow v \in x)$. By contrast, $\phi^L = (\forall v \in L)(v \in z \Rightarrow v \in x)$ where z and x are parameters representing elements in L. It is obvious that if ϕ is true in V, then ϕ^L is likewise true. Conversely, suppose ϕ^L is true in V, which is the same as saying that ϕ is true in L. Since L is transitive, $z \subseteq x$ is equivalent to $z \in x$, hence $(v \in z \Rightarrow v \in x)$. So ϕ is true in V for parameters $x, z \in L$.

This shows that $\subseteq$ is absolute on any transitive domain. That means that subsets are the same in L as in V. Consequently, the power set axiom is true in L.

It has already been shown (Lemma 11.21) that if ϕ is the formula for Axiom A1, then ϕ^L is provable from ZF, because L is a transitive class.

Next, the Axiom of Foundation is true in L: Indeed, suppose $a \in L$. Recall that $L \subseteq V$. Now, since the Axiom of Foundation is true in V, there is an element $b \in V$ such that $(b \in a) \wedge (b \cap a =\varnothing)$. Since L is transitive, it follows from $b \in a$ and $a \in L$ that $b \in L$. Thus, the following holds in L: $(b \in a) \wedge (b \cap a =\varnothing)$. From Theorem 11.5(d), L is well-founded.

Several of the ZF axioms have now been shown to hold in L; others are very technical and require more elaborate machinery for their proof. They are omitted here. ■

It turns out to be an important fact that the Axiom of Constructibility (V = L) holds in L. You may recall that the Axiom of Constructibility has been added to ZF, and it might appear that, as a consequence, it is true in L, since L is a model of ZF. However, as mentioned above, what we need to establish is that $(V = L)^L$ holds in L—and that turns out to be more difficult than it appears to be.

The Axiom of Constructibility is the claim that every set is constructible, in other words it is the formula $(\forall x)(\exists \alpha)[On(\alpha) \wedge x \in L_\alpha]$ where $On(\alpha)$ stands for "α is an ordinal". As explained previously, what we really need to show is that it is the formula for $(V = L)^L$ that holds in L. Now $(V = L)^L$ is the formula

$$(\forall x \in L)(\exists \alpha \in L)[On(\alpha) \wedge (x \in L_\alpha)^L]$$

It can be shown that the formulas $On(\alpha)$ as well as $(x \in L_\alpha)^L$ are absolute in L. The proofs are fairly technical, and omitted here. Using these facts, it follows that $(V = L)^L$ is absolute, and therefore is true in the model L. Our conclusion is:

11.26 Theorem*L* is a model of ZF + (V = L).

Consequently, the Axiom of Constructibility V = L cannot be refuted in ZF, so it may be added as a new axiom without danger of contradiction. From this point onward, we assume that V = L is one of our axioms.

We are now ready to reap the whirlwind!

EXERCISES 11.5

1. For any class A, prove each of the following:

a) If A is transitive, then $A \subseteq \mathscr{P}^*(A)$.

b) If $X \subseteq A$ and X is finite, then $X \in \mathscr{P}^*(A)$. If A is an infinite set, then $|\mathscr{P}^*(A)| = |A|$.

2. Prove that for every ordinal α, $L_\alpha \cap On = \alpha$, where On is the class of the ordinals.
3. For all finite n, prove that $L_n = V_n$.
4. For all infinite cardinals α, prove that $|L_\alpha| = |\alpha|$.
5. Of all the axioms of set theory (including AF, V = L, and AC), which are true in the empty set? Which are true in the finite model A whose elements are: a, b, $\{a\}$, $\{b\}$, $\{a, b\}$?
6. Referring to the previous exercise, let B be the subset of A whose elements are a, b. Write a formula ϕ such that ϕ is true in A but ϕ^B is not true in A.
7. Explain why each of the following formulas is absolute over transitive domains:

   a) $a \subseteq b$.

   b) $c = (a, b)$ where (a, b) denotes the ordered pair.

   c) $c = a \cup b$.

   e) $c = a \cap b$.

   f) $c = a \cup \{a\}$.

   g) a is a transitive set.
8. It was mentioned in Chapter 1 that there is an axiom proposed by von Neumann (but not used here) that states the following: A is a proper class iff A is in 1-1 correspondence with V. Equivalently: A is a set iff A is not in 1-1 correspondence with V. Call this axiom VN:

   a) Prove that VN implies the Axiom of Choice.

   b) Explain why VN implies a strengthened version of the Axiom of Choice that applies not only to sets but to all classes. What does this do to the well-ordering theorem?

   c) Prove that VN implies the Axiom of Replacement (our Axiom A9).

   d) Prove that Axiom A9 together with the strengthened Axiom of Choice imply VN.

6 THE GÖDEL THEOREMS

Using ZF + (V=L), Gödel was able, first, to show that the Axiom of Choice is true in L. He did not show this directly by proving that there are choice functions. Rather, he proved that it was possible to construct a well-ordering of L, from which it is obvious that every set can be well-ordered because every set is a subset of L. And as we know, the well-ordering theorem is equivalent to the Axiom of Choice.

11.27 Theorem If V=L, then the Axiom of Choice holds. This implies that if ZF is consistent, then $ZF \nvdash \neg AC$.

Proof. The proof consists of defining a constructible relation which is an order relation $\prec$ on L, and showing that $\prec$ well-orders L. Let x and y be two arbitrary elements of L, and let us define the condition which makes $x \prec y$ true. There are three cases, treated differently:

1. Let $\alpha < \beta$ be two cardinals such that x first appears in L_α, whereas y first appears in L_β. We then decree that $x \prec y$ in the ordering of L we are defining.

2. Now suppose that x and y have their first occurrence in the same L_y, call it $L_{\alpha+1}$. For this case, let $\{\phi_i : i < \omega\}$ be an enumeration of all the formulas of our countable language. We reason by induction, so we assume that the ordering $\prec$ has already been defined on L_α and well-orders it. Suppose x is defined by a formula ϕ_x and y is defined by a formula ϕ_y, where both formulas have (for simplicity) just one parameter—say p in ϕ_x and r in ϕ_y:

$$x = \{q \in L_\alpha : L_\alpha \vDash \phi_x(q, p)\} \text{ and } y = \{q \in L_\alpha : L_\alpha \vDash \phi_y(q, r)\}$$

We decree that $x \prec y$ if ϕ_x precedes ϕ_y in the enumeration of formulas. Or, in case $\phi_x = \phi_y$, then $x \prec y$ if p precedes r in the order relation on L_α.

Recall that the class of the ordinal number is well-ordered, and by induction on α, L_β is well-ordered by $\prec$ for every $\beta < \alpha$. Using these facts, the order relation $\prec$ we have just defined well-orders L. ■

11.28 Theorem$L$ satisfies CH (the Continuum Hypothesis), that is, $ZF \vdash (CH)^L$. It follows that if ZF is consistent then $ZF \nvdash \neg CH$.

The technical machinery required to show that CH is true in L goes well beyond the scope of this book. We shall therefore confine ourselves here to outlining the intuitive basis for the proof. We have already discussed the implications of the fact that L contains only constructible sets: For $\alpha< \omega_1$, (where ω_1 is the first uncountable ordinal) L_α is countable: Every set in L_α is defined by a formula ϕ in the countable language of set theory. Moreover, the language includes no more than countably many parameters, because the parameters denote elements in the countable set L_α. So there are only countably many formulas.

Since there are no more than countably many formulas available to construct sets, there are no more than countably many sets in L_α for each $\alpha< \omega_1$. So finally, the union of a strictly increasing family of $\omega_1 = \aleph_1$ many countable sets has cardinality no less than $\aleph_1$ nor greater than $\aleph_1$. A formal version of this proof can be generalized to higher cardinals, and yields the following theorem:

11.29 Theorem$L$ satisfies GCH (the Generalized Continuum Hypothesis), $2^{\aleph_\alpha} = (\aleph_{\alpha+1})^L$. Consequently, $ZF \vdash (GCH)^L$.

For a brief recapitulation of what has been done, we have shown that if you assume ZF to be consistent, then so is ZF + AF + V = L + AC + GCH. After Gödel demonstrated that the Axiom of Choice and the Continuum Hypothesis are relatively consistent with the ZF axioms—that is, if we assume the ZF system is consistent, then so are AC and GCH—the question was immediately raised whether the negations of AC and GCH are likewise consistent with ZF. If they are, that means that AC and GCH are *independent of* the ZF axioms—in other words, they may be added to the ZF axioms without contradiction, and likewise, their negations may be added to ZF without producing contradictions.

It was not until 1964 that this question was answered affirmatively: In a groundbreaking paper, Paul Cohen invented a method called *forcing* to show that indeed (if ZF is consistent) it is possible to construct models of ZF that satisfy $\neg AC$ and $\neg GCH$. Perhaps these results do not advance the practice of mathematics to any appreciable degree, but their philosophical impact is tremendous. They show that the concept of set—though founded on very concrete intuitions and mental pictures—is not nearly as elementary as we are prepared to believe. There is no one unique truth concerning the properties of sets: Rather, the reality of what a set is bifurcates into several alternative realities, all equally plausible and all equally true.

Lastly, it has been shown that we will never have absolute certainty that set theory—or mathematics generally—is free of contradictions. It is not merely a question of the state of current knowledge: Rather, what has been shown is that it is *fundamentally* impossible ever to prove the consistency of mathematics.

For set theory, is that really surprising? Think of it! It is already a remarkable fact that animals (including homo sapiens) are able to abstract out of experience the fact that there is such a thing as a cluster, a batch, a bundle of similar objects—and that such a bundle is *a separate unit of reality*. Then we build on that and think of collections we've never experienced, such as all the trees in a forest, or all the natural numbers. Then we abstract further and think of a *set* as a thing in itself, irrespective of what its members are. The notion of *set* is the abstraction of an abstraction of an abstraction. That kind of iterated abstracting seems to be the essence of the human intellectual enterprise. More than that, it is perhaps the long-term ecological function of brains.

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